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We explore two facets of topology, coarse and computational, that share a similar philosophy: "The perceived shape of a space depends on the scale at which that space is viewed". In coarse topology, we analyze the preservation of properties of coarse spaces over direct products. We define a free product of a coarse space and prove these properties are preserved under this operation. On the computational side, we present a new class of topological descriptors called Persistence Curves. We prove the stability of these descriptors, we show that they generalize another popular descriptor called Persistence Landscapes, and finally we use these persistence curves to perform texture classification with great results on four popular texture databases: Outex, UIUCTex, KTH-TIPS, and FMD.

ON THE PRESERVATION OF COARSE PROPERTIES OVER PRODUCTS AND ON PERSISTENCE CURVES

by

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Approved by

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To my mother and father.

APPROVAL PAGE

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PREFACE

Topology can be loosely defined as the study of shape. This field of mathematics examines both quantitative properties of a space, such as dimension or number of holes, and qualitative properties such as connectedness or compactness. In this work, we explore two sub-fields of topology: Coarse Geometry and Computational Topology. Coarse Geometry finds a home in what is commonly referred to as pure mathematics while Computational Topology tends to lie more in applied mathematics. While these two fields lie at opposite ends of this spectrum, they share and are motivated by the same philosophy: "The shape of a space is dependent upon the scale at which it is viewed."

Consider the integers as a subset of the reals. One can imagine that by "zooming in" close enough, the integer line looks just like a single point. As we begin to step back, our view of the space changes and we are presented with a sequence of evenly spaced points. We keep stepping back only to find the space between the points seems to grow smaller. At this point, we are beginning to blur the space and lose sight of its small-scale properties. By stepping back even farther, the space looks indistinguishable from the real line, and we begin seeing the large scale properties. This change in perspective is at the heart of Coarse Geometry. Here, we concern ourselves with so-called coarse structures. These structures generalize the idea of a metric by replacing distance with a notion of proximity. These structures allow us to study large-scale properties of spaces. In Chapter I we examine to what extent these large-scale properties are preserved by direct products. In Chapter II, we introduce a free product for these coarse spaces and show that these properties are invariant under this operation.

Consider the integers once more. We will alter the scale in a different way. Place a small disk around each point and allow each disk to grow. As these disks grow they intersect and we place segments between the centers of the disks. Again, the result looks like the real line. Here, we find ourselves in the field of Computational Topology. The motivation for this area of study was accurately summed up by Gunnar Carlsson: “Data have shape and shape has meaning.” In Computational Topology, we are most often concerned with applying the methods of Algebraic Topology to real-world data sets. This is often done through the use of a tool called Persistent Homology, which considers the data set at all scales and produces a visual summary called a persistence diagram. Recently, there has been a large push to combine Computational Topology with Machine Learning, often through vectorization of persistence diagrams. In Chapter III, we propose a stable vectorization of these diagrams and examine their performance on a variety of texture datasets. We also show this class of curves generalizes several other popular vectorizations such as the Persistence Landscapes, Entropy Summary function, Euler Characteristic Curve, Persistent Homology Transform, and Euler Characteristic Transform.

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CHAPTER I

COARSE DIRECT PRODUCTS

We begin this chapter with a brief discussion of metric spaces.

I.1. Metric Spaces

The large-scale approach to metric spaces was described first by Gromov [Gro93]. Metric spaces are sets equipped with a notion of distance. They serve as a motivation for a natural generalization discussed in Section I.2.

Definition I.1. Let X be a set and let $d : X \times X \to [0, \infty)$. The pair (X, d) is called a **metric space** if:

- $d(x, y) = 0$ if and only if $x = y$;
- for every $x, y \in X$, $d(x, y) = d(y, x)$; and
- for every $x, y, z \in X$,$d(x, z) \leq d(x, y) + d(y, z)$.

The last bullet is referred to as the **triangle inequality**. By convention, if U, V are subsets of a metric space (X, d), we define $d(U, V) := \inf\{d(u, v) \mid u \in U, v \in V\}$. The map d is called a **metric**.

Definition I.2. A metric space (X, d) is called a **discrete metric space** if $\operatorname{im} d$ is countable and $\inf\{d(x, y) \mid x, y \in X, x \neq y\} > 0$

For the large-scale approach to metric spaces, we're interested in asymptotic behavior. This contrasts the general idea of metric space topology where we are concerned with ε scale where ε is considered to be small, we consider R-disjoint where R is considered to be large.

Definition I.3. Let (X, d) be a metric space. Suppose $\mathcal{U}$ is a family of subsets of X. Let $R > 0$ be a real number. We say $\mathcal{U}$ is **R-disjoint** if whenever U and V are distinct elements of $\mathcal{U}$ we have $d(U, V) \geq R$. We say $\mathcal{U}$ is **uniformly bounded** if $\sup\{\operatorname{diam}(U) \mid U \in \mathcal{U}\} < \infty$

When we think about a space, we often would like to know its dimension. This word has many meanings in mathematics. The next definition is the large-scale analog to the standard Lebesgue covering dimension of metric spaces.

Definition I.4. [Gro93] A metric space (X, d) is said to have **asymptotic dimension of at most** n, denoted $\operatorname{asdim} X \leq n$ if for every positive real number R there are $n+1$ families $\mathcal{U}_0, \ldots \mathcal{U}_n$ of subsets of X so that

- $\cup_{i=0}^{n} \mathcal{U}_i$ forms a cover of X;
- each $\mathcal{U}_i$ is R-disjoint; and
- each $\mathcal{U}_i$ is uniformly bounded.

It is not the case that every metric space is of finite asymptotic dimension. However, metric spaces of infinite asymptotic dimension may still exhibit interesting properties. The following property was introduced by Dranishnikov [Dra00] as a large-scale analog to Haver's property C.

Definition I.5. A metric space (X, d) is said to have **asymptotic property C** if for every sequence of positive real numbers $R_1, R_2, \ldots$ there is some finite sequence of families $\mathcal{U}_1, \ldots \mathcal{U}_n$ of subsets of X so that

- $\cup_{i=0}^{n} \mathcal{U}_i$ forms a cover of X;
- each $\mathcal{U}_i$ is R_i-disjoint; and
- each $\mathcal{U}_i$ is uniformly bounded.

One can think about dimension as a notion of complexity. As in Linear Algebra, dimension can aid in understanding how a space breaks down or decomposes. We present a notion of decomposition and complexity on the large-scale here.

Definition I.6. [] Let (X, d) be a metric space. Suppose $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X. Suppose R is a positive real number and n is a positive integer. We say $\mathcal{U}$ (R, n)**-decomposes** over $\mathcal{V}$ if for each $U \in \mathcal{U}$ there are n subfamilies of $\mathcal{V}$, $\mathcal{V}_1, \ldots \mathcal{V}_n$ so that

- each $\mathcal{V}_i$ is R-disjoint and
- $\cup_{i=1}^{n} \mathcal{V}_i$ covers U.

Definition I.7. [] A metric space (X, d) has **straight finite decomposition complexity** if for every sequence of real numbers $R_0, R_1, \ldots$ there is a finite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots \mathcal{U}_n$ of subfamilies of X so that

- $\mathcal{U}_0 = \{X\}$;
- each $\mathcal{U}_i$ $(R_i, 2)$-decomposes over $\mathcal{U}_{i+1}$;
- $\mathcal{U}_n$ is uniformly bounded.

Definition I.8. [] A discrete metric space (X, d) is said to have **property A** if for each $\varepsilon > 0$ and for each $R > 0$ there is a family of finite sets $\{A_x\}_{x \in X}$ where $A_x \subseteq X \times \mathbb{N}$ so that

- $(x, 1) \in A_x$ for each $x \in X$;
- if $d(x, y) < R$, then $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$; and
- there is an $S > 0$ so that whenever $(y, n) \in A_x$, $d(x, y) < S$.

We end this section with a brief discussion of trees as metric spaces.

Definition I.9. An **undirected graph** is a set of points V called **vertices** along with a set E of unordered pairs of vertices called **edges**. A **path** from $a \in V$ to $b \in V$ is a sequence of distinct points $a = x_0, x_1, \ldots x_n = b$ of V so that $(x_i, x_{i+1}) \in E$ for each $i = 0, \ldots n$. Finally, a **tree** $T = (V, E)$ is an undirected graph where between any two vertices there is a unique path of distinct points.

We can realize a tree as a metric space via the **path length metric**. In this metric, the distance between vertices a and b is the number of edges in the unique path between them.

I.2. Coarse Spaces

In this section, we will develop a generalization of metric spaces to a large-scale notion often referred to as the coarse category. Instead of metric spaces, we will work with so-called coarse spaces, which focus on a notion of proximity (or closeness) instead of a metric. The coarse category was defined by Roe [Roe03].

Definition I.10. Let X be a set and let $\mathcal{E}$ be a collection of subsets of $X \times X$. We call the tuple $(X, \mathcal{E})$ a **coarse space** if the following conditions hold:

1) the **diagonal** $\Delta = \{(x, x) \colon x \in X\}$ is in $\mathcal{E}$;

2) if $E \in \mathcal{E}$ and $F \subseteq E$ then $F \in \mathcal{E}$;

3) if $E, F \in \mathcal{E}$ then $E \cup F \in \mathcal{E}$;

4) if $E, F \in \mathcal{E}$ then the **composition** $E \circ F = \{(x, z) \colon \exists y, (x, y) \in E \;\&\; (y, z) \in F\}$ is in E; and

5) if $E \in \mathcal{E}$ then the **inverse** $E^{-1} = \{(y, x) \colon (x, y) \in E\}$ is in $\mathcal{E}$.

In the instance that $(X, \mathcal{E})$ is a coarse space, we call elements of $\mathcal{E}$ **entourages**.

We often think of an entourage as a designation of closeness. For example if E is an entourage and $(x, y) \in E$ we think that x is close to y. One could also think of these pairs as starting and stopping points for some process. With either of these concepts in mind, the definition is intuitive. To ease notation, we will often identify a coarse space $(X, \mathcal{E})$ by X when the context is clear. We present some common examples of coarse structures below.

Example I.11. The trivial coarse structure contains only the diagonal and its subsets.

Example I.12. If X is a set, the **discrete coarse structure** is formed by taking $\mathcal{E}$ to be the collection of sets that have at most finitely many off-diagonal points, see Figure I.1.

Example I.13. Let (X, d) be a metric space. For $R > 0$ define the set $E_R = \{(x, y) \in X \times X \colon d(x, y) \leq R\}$. We take $\mathcal{E}$ to be the subset closure of $\{E_R \colon R > 0\}$. This forms a coarse structure called the **bounded coarse structure** on the metric space X.

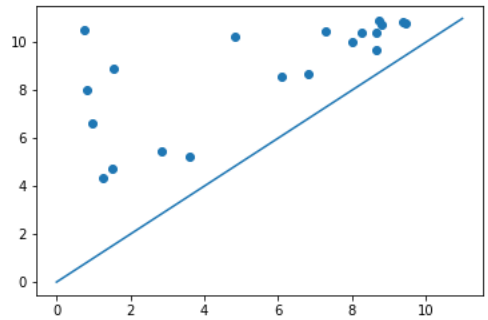


Figure I.1. An Entourage in the Discrete Coarse Structure

Example I.14. [Wri02] If (X, d) is a metric space, the C_0 **coarse structure** consists of all subsets E of $X \times X$ so that for all $\varepsilon > 0$ there is some compact set $K \subseteq X$ for which $d(x, y) < \varepsilon$ if $(x, y) \in E \setminus (K \times K)$.

Example I.15. Let $\mathcal{E}_0(X)$ represent the C_0 coarse structure on the metric space X. Because compact subsets of the integers $\mathbb{Z}$ are exactly the finite sets, we see $\mathcal{E}_0(\mathbb{Z})$ is the discrete coarse structure on $\mathbb{Z}$.

Example I.16. Let $\mathcal{B}(X)$ represent the bounded coarse structure on the metric space X. We see $\mathcal{E}_0(X) \subseteq \mathcal{B}(X)$. We saw in the previous example that $\mathcal{E}_0(\mathbb{Z})$ is exactly the discrete coarse structure on $\mathbb{Z}$ Consider the $\mathbb{Z}$ equipped with the bounded coarse structure associated to the usual metric $\mathcal{B}(\mathbb{Z}$. The set $\{(x, x+1) \mid x \in \mathbb{Z}\}$ is an infinite set that belongs to $\mathcal{B}(\mathbb{Z})$. Hence, the containment $\mathcal{E}_0(X) \subseteq \mathcal{B}(X)$ can be strict.

Definition I.17. Let $\mathcal{U}$ be a family of subsets of a coarse space X and let E be an entourage of X.

- If $B \subseteq X$ we say B is **bounded** if $B \times B$ is an entourage.
- We say $\mathcal{U}$ is E**-disjoint** if whenever $U, U' \in \mathcal{U}$ and $U \neq U'$ we have $(U \times U) \cap E = \emptyset$.
- We call $\mathcal{U}$ **uniformly bounded** if $\bigcup_{U \in \mathcal{U}} U \times U$ is an entourage.

Proposition I.18. *Let $\mathcal{U}$ be a family of subsets of a coarse space X. Further suppose that $\mathcal{U}$ is E-disjoint for some entourage E. If $F \subseteq E$ then $\mathcal{U}$ is F disjoint.*

□

Definition I.19. Let $(X, \mathcal{E})$ be a coarse space and let $A \subseteq X$. For $E \in \mathcal{E}$ we define the E**-ball** around A to be the set $E[A] = \{x \in X : \exists y \in A, (x, y) \in E\}$. This set is an analog of a ball in a metric space.

Definition I.20. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. Let $f : X \to Y$ be a map.

1. The map f is called **proper** if $f^{-1}(A)$ is bounded in X whenever $A \subseteq Y$ is bounded in Y.
2. The map f is called **uniformly expansive** (also sometimes called **bornologous** or **coarsely expansive**) if $(f \times f)(E) = \{(f(x), f(x')) : (x, x') \in E\} \in \mathcal{F}$ whenever $E \in \mathcal{E}$.
3. The map f is said to be a **coarse map** if it is both bornologous and proper.
4. The map f is called a **coarsely uniform embedding** if it is a coarse map and $(f \times f)^{-1}(F) = \{f^{-1}(y) \times f^{-1}(y') : (y, y') \in F\} \in \mathcal{E}$ whenever $F \in \mathcal{F}$.

5. Two maps f and f' from X to Y are said to be **close** if $\{(f(x), f'(x))\colon x \in X\} \in \mathcal{F}$.

Definition I.21. Two coarse spaces X and Y are **coarsely equivalent** if there exist coarse maps $f : X \to Y$ and $g : Y \to X$ so that $f \circ g$ is close to id_Y and $g \circ f$ is close to id_X. In such a case, the maps f and g are called **coarse equivalences**.

Definition I.22. Let $(X, \mathcal{E})$ be a coarse space. We say X has **coarse asymptotic dimension of at most** n and write $\operatorname{asdim}_C(X) \leq n$ if for any $E \in \mathcal{E}$ there exist families $\mathcal{U}_0, \mathcal{U}_1, \ldots \mathcal{U}_n$ of subsets of X so that

1) The collection $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ forms a cover of X

2) For $i = 0, 1, \ldots, n$, the family $\mathcal{U}_i$ is E-disjoint.

3) For $i = 0, 1, \ldots, n$, the family $\mathcal{U}_i$ is uniformly bounded.

Proposition I.23. *Coarse asymptotic dimension passes to subsets. That is if Y is a subset of a space X and if* $\operatorname{asdim} X \leq n$ *then* $\operatorname{asdim} Y \leq n$.□

Example I.24. We have $\operatorname{asdim}_C(\mathbb{R}) \leq 1$ where $\mathbb{R}$ has the bounded coarse structure over the usual distance. Indeed given some entourage L there is some $R > 0$ so that $(x, y) \in L$ implies $d(x, y) \leq R$. Then we take $\mathcal{U}_0 = \{[2kR, (2k+1)R]\colon k \in \mathbb{Z}\}$ and $\mathcal{U}_1 = \{[(2k-1)R, (2k)R]\colon k \in \mathbb{Z}\}$. One can quickly see the desired properties hold. Figure I.2 shows $\operatorname{asdim}_C(\mathbb{R}^2) \leq 2$.

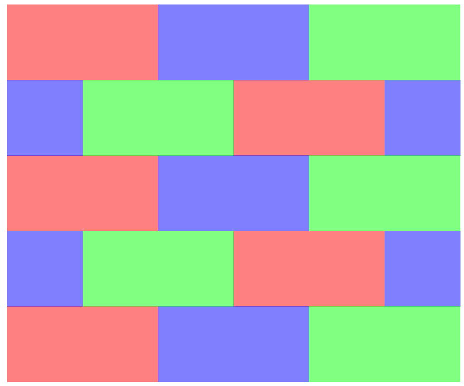


Figure I.2. Pictoral Proof of asdim$(\mathbb{R}^2) \leq 2$

Example I.25. [BD01] Trees with the bounded coarse structure corresponding to the path-length metric have coarse asymptotic dimension of at most 1. Indeed we fix a vertex x_0 of the tree. Then, given an entourage L we can find the associated $R > 0$ and consider the sets $A_k = \{a \in T \mid kR \leq d(x_0, a) < (k+1)R\}$. Clearly this collection of sets cover the tree. However, we note these sets are not uniformly bounded. To fix this we further divide each set by defining relations on them. For $a, b \in A_k$ we say $a \sim b$ if there is a point z contained in both the path from x_0 to a and x_0 to b so that $d(x_0, z) \geq (k - \frac{1}{2})R$. We see this relation is obviously reflexive and symmetric. Suppose $a \sim b$ and $b \sim c$. Then there is some z_0 in the common path from x_0 to a and to b with $d(x_0, z_0) \geq (k - \frac{1}{2})R$. Similarly there is a z_1 on the common path from x_0 to b and to c with $d(x_0, z_1) \geq (k = \frac{1}{2}R)$. Since the path from x_0 to b is unique, and since both z_0 and z_1 are on the path, we simply take the vertex that appears first in the path to see $a \sim c$. Hence we have an equivalence relation. We consider the resulting partition, $\mathcal{A}_k$ of the set A_k. Suppose $B \in \mathcal{A}_k$ and let $a, b \in B$. Take the corresponding z from $\sim$. Then $d(a, b) \leq d(x_0, a) - d(x_0, z) + d(x_0, b) - d(x_0, z) \leq (k+1)R + (k+1)R - (2k-1)R = 3R$. Finally, by taking $\mathcal{U} = \mathcal{A}_0 \cup \mathcal{A}_2 \cup \ldots$ and $\mathcal{V} = \mathcal{A}_1 \cup \mathcal{A}_3 \cup \ldots$ we obtain two R-disjoint (hence L-disjoint) uniformly bounded families that cover T.

Proposition I.26. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. Suppose* $\operatorname{asdim}_C(X) \leq n$ *and suppose $f : Y \to X$ is uniformly expansive and proper, then* $\operatorname{asdim}_C(Y) \leq n$.

Proof. Let $f : Y \to X$ be uniformly expansive and proper. Let $F \in \mathcal{F}$ be given. We must find families $\mathcal{V}_0, \ldots, \mathcal{V}_n$ that cover Y, are F-disjoint and are uniformly bounded. To this end consider the fact that $E := (f \times f)(F) \in \mathcal{E}$. Use the assumption that $\operatorname{asdim}_C(X) \leq n$ to produce families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ that cover X, are E-disjoint, and are uniformly bounded. We define $\mathcal{V}_i := \{f^{-1}(U) \mid U \in \mathcal{U}_i\}$ for $i = 0, \ldots n$. It is clear that because the families $\mathcal{U}_i$ all together cover X, the families $\mathcal{V}_i$ all together cover Y. Moreover, if $V \in \mathcal{V}_i, V' \in \mathcal{V}_i$, and $V \neq V'$ then there are $U, U' \in \mathcal{U}_i$ with $U \neq U'$ so that

$$\begin{aligned}
V \times V' \cap F &= (f \times f)^{-1}(U \times U') \cap F \\
&\subseteq (f \times f)^{-1}(U \times U') \cap (f \times f)^{-1}(E) \\
&= (f \times f)^{-1}(U \times U \cap E) \\
&= (f \times f)^{-1}(\emptyset) \\
&= \emptyset
\end{aligned}$$

Hence, each $\mathcal{V}_i$ is F-disjoint. Finally we show that each $\mathcal{V}_i$ is uniformly bounded. To this end consider

$$\bigcup_{V \in \mathcal{V}_i} V \times V = \bigcup_{U \in \mathcal{U}_i} (f \times f)^{-1}(U \times U) = (f \times f)^{-1} \left(\bigcup_{U \in \mathcal{U}_i} U \times U \right)$$

Since $\cup_{U \in \mathcal{U}_i}(U \times U) \in \mathcal{E}$ and f is proper, we see $(f \times f)^{-1} \left(\bigcup_{U \in \mathcal{U}_i} U \times U \right)$ hence $\mathcal{V}_i$ is uniformly bounded. Therefore, since we have found families $\mathcal{V}_0, \ldots \mathcal{V}_n$ that cover Y, are F-disjoint and uniformly bounded, we conclude $\operatorname{asdim}_C Y \leq n$. □

It is apparent that not every coarse space has finite coarse asymptotic dimension. Consider the example below.

Example I.27. Consider the set $\bigoplus \mathbb{Z}$ of integer sequences of only finitely many nonzero elements. We can impose a metric on this set as follows, $d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} n \cdot |x_n - y_n|$. It is clear then that $\mathbb{Z}^n$ can be coarsely embedded into this space. Thus since asymptotic dimension is monotonic, we see that for each $n \in \mathbb{N}$, $\operatorname{asdim} \bigoplus \mathbb{Z} > n$, i.e. $\operatorname{asdim} \bigoplus \mathbb{Z} = \infty$

Even though there are spaces with infinite asymptotic dimension, we can measure to what extent they have ininite dimension..

Definition I.28. A coarse space X is said to have **coarse property C** if for any sequence $E_0, E_1, \ldots$ of entourages, there exist families $\mathcal{U}_0, \ldots \mathcal{U}_n$ so that

- $\mathcal{U}_0 \cup \ldots \cup \mathcal{U}_n$ forms a cover of X;
- each $\mathcal{U}_i$ is E_i-disjoint; and
- each $\mathcal{U}_i$ is uniformly bounded.

Proposition I.29. *Coarse property C passes to subsets* □

In the example below, we see that every space with finite asymptotic dimension also has coarse property C.

Example I.30. Suppose $\operatorname{asdim}_C(X) \leq n$ and let $E_0, E_1, \ldots$ be a sequence of entourages. Then by applying the definition of coarse asymptotic dimension to the set $\bigcup_{i=0}^{n} E_i$ we immediately see that X has coarse property C.

Example I.31. [Yam15] In a proper metric, $\bigoplus \mathbb{Z}$ has asymptotic (hence coarse) property C.

Proposition I.32. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and suppose X has coarse property C. If $f : Y \to X$ is uniformly expansive and proper then Y has coarse property C.*

Definition I.33. A coarse space $(X, \mathcal{E})$ will be said to have **coarse property A** if for each $\varepsilon > 0$ and for each $E \in \mathcal{E}$ there exists a map $a : X \to \ell^1(X)$, expressed as $a : x \mapsto a_x$ such that:

1. $\|a_x\|_1 = 1$ for all $x \in X$;
2. if $(x, y) \in E$, then $\|a_x - a_y\|_1 < \varepsilon$;
3. there exists $S \in \mathcal{E}$ such that for each $x \in X$, $\operatorname{supp} a_x \subseteq S[x]$.

Proposition I.34. *Property A passes to subsets.*

Proof. Indeed if $Y \subseteq X$ and X has property A then for $y \in Y$ we consider the map $a_y : X \to \ell^1(X)$ restricted to Y, denoted by a_y^Y. We are not guaranteed that $\|a_y^Y\| = 1$, however we do know $\|a_y^Y\| < \infty$ therefore the map $\frac{a_y^Y}{\|a_y^Y\|}$ has the desired properties. $\square$

Theorem I.35. *[BMN17] Let $(X, \mathcal{E})$ be a coarse space. If X has coarse property C, then X has coarse property A.*

Corollary I.36. *If X has finite asymptotic dimension then X has coarse property A.*

Definition I.37. Let Y be a subset of a coarse space $(X, \mathcal{E})$ and let $\mathcal{U}$ be a family of subsets of X. Let n be a positive integer and let $E \in \mathcal{E}$ be an entourage. We say that Y admits an (E, n)-**decomposition** over $\mathcal{U}$ if Y can be expressed as a union of n sets $Y^1, Y^2, \ldots, Y^n$ in such a way that each Y^i can be expressed as an E-disjoint union of sets from $\mathcal{U}$. Here, by an E-**disjoint union of sets from** $\mathcal{U}$, we mean that each $Y^i = \sqcup_j Y_j^i$, where $Y_j^i \times Y_{j'}^i \cap E = \emptyset$ if $j \neq j'$ and $Y_j^i \in \mathcal{U}$ for all j.

Definition I.38. The coarse space $(X, \mathcal{E})$ is said to have **straight finite coarse decomposition complexity (sFCDC)** if for any sequence of entourages $E_1, E_2, ...$ there is a sequence of families $\mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_n$ so that

1. $\mathcal{U}_0 = \{X\}$;
2. for every i, each $U \in \mathcal{U}_i$ admits an $(L_i, 2)$-decomposition over $\mathcal{U}_{i+1}$; and
3. $\mathcal{U}_n$ is uniformly bounded.

Proposition I.39. *sFCDC passes to subsets.*

Proposition I.40. *If X and Y are coarsely equivalent and X has sFCDC, then Y has sFCDC.*

Theorem I.41. *[BMN17] Let $(X, \mathcal{E})$ be a coarse space. If X has sFCDC then X has property A.*

Theorem I.42. *[BMN17] Let $(X, \mathcal{E})$ be a coarse space. If X has coarse property C then X has sFCDC.*

Definition I.43. For each $i = 1, 2, \ldots, k$, suppose that $(X_i, \mathcal{E}_i)$ is a coarse space. We denote by p_i the projection map $p_i : X_1 \times X_2 \times \cdots \times X_k \to X_i$ from the product. We define the **product coarse structure** on the product $X_1 \times \cdots \times X_k$ by

$$\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_k = \left\{ E \subseteq (X_1 \times \cdots \times X_k)^2 \colon (p_i \times p_i)(E) \in \mathcal{E}_i \text{ for each } i \in \{1, 2, \ldots, k\} \right\}.$$

The example below shows that a coarse structure on the product of two spaces does not necessarily correspond to the coarse product of those same types of coarse structure on the factors.

Example I.44. Let $I = [0, 1]$. Consider the coarse spaces $((\mathbb{Z} \times I), (\mathcal{E}_0(\mathbb{Z}) \otimes \mathcal{E}_0(I)),$ and $((\mathbb{Z} \times I), \mathcal{E}_0(\mathbb{Z} \times I))$. Consider the set $E = \cup_{n \in \mathbb{Z}}(\{n\} \times I) \times (\{n\} \times I)$. We see $(p_1 \times p_1)(E) = \Delta_{\mathbb{Z}}$ and $(p_2 \times p_2)(E) = I$. Thus $E \in \mathcal{E}_0(\mathbb{Z}) \otimes \mathcal{E}_0(I)$. Now take $\varepsilon = \frac{1}{2}$. Because compact sets of $\mathbb{Z} \times I$ contain only finitely many distinct first coordinates we see that for any compact set K there are points of $E \setminus (K \times K)$ of distance 1. That is to say that $E \notin \mathcal{E}_0(\mathbb{Z} \times I)$.

By contrast, we see that the bounded coarse structure is stable over coarse products.

Example I.45. For any two metric spaces X and Y, $\mathcal{B}(X \times Y) = \mathcal{B}(X) \otimes \mathcal{B}(Y)$.

I.3. Products and Coarse Property A and Coarse Property C

In this section, we show that coarse property A and coarse property C are preserved by the coarse direct product.

Theorem I.46. *If $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ have coarse property A, then $(X \times Y, \mathcal{E} * \mathcal{F})$ has coarse property A.*

Proof. Let $L \in \mathcal{E} * \mathcal{F}$ and $\varepsilon > 0$ be given. Apply property A to X for the set $E = (p_X \times p_X)(L)$ and $\varepsilon/2$ to obtain a map $a^X : X \to \ell^1(X)$ and a set $S \in \mathcal{E}$ so that $\|a^X_x\| = 1$ for every $x \in X$, $\|a^X_x - a^X_{x'}\| \leq \varepsilon/2$ whenever $(x, x') \in E$ and $\operatorname{supp}(a^X_x) \subseteq S[x]$ for every $x \in X$. Similarly, apply property A to Y for the set $F = (p_Y \times p_Y)(L)$ and $\varepsilon/2$ to obtain a map $a^Y : Y \to \ell^1(Y)$ and a set $T \in \mathcal{F}$ so that $\|a^Y_y\| = 1$ for every $y \in Y$, $\|a^Y_y - a^Y_{y'}\| \leq \varepsilon/2$ whenever $(y, y') \in F$ and $\operatorname{supp}(a^Y_y) \subseteq T[y]$ for every $y \in Y$.

Define a new map $a : (X \times Y) \to \ell^1(X \times Y)$ by $a(x,y) = a_{(x,y)} = a^X_x a^Y_y$. Clearly $\|a_{(x,y)}\| = 1$ for all $(x,y) \in X \times Y$. Moreover, if $(x,y,x',y') \in L$ we have $(x,x') \in E, (y,y') \in F$ and

$$\begin{aligned}\|a_{(x,y)} - a_{(x',y')}\| &= \|a^X_x a^Y_y - a^X_{x'} a^Y_{y'}\| \\ &\leq \|a^X_x a^Y_y - a^X_x a^Y_{y'}\| + \|a^X_x a^Y_{y'} - a^X_{x'} a^Y_{y'}\| \\ &= \|a^X_x\| \|a^Y_y - a^Y_{y'}\| + \|a^Y_{y'}\| \|a^X_x - a^X_{x'}\| \\ &\leq \varepsilon\end{aligned}$$

Finally, define $U = \{(s_1, t_t, s_2, t_2) \mid (s_1, s_2) \in S, (t_1, t_2) \in T\}$. Then $U \in \mathcal{E} * \mathcal{F}$, and if $a_{(x,y)}(z,w) \neq 0$ then $a^X_x(z) \neq 0$ and $a^Y_y(w) \neq 0$. Hence $z \in S[x]$, $w \in T[y]$, and so $(z,w,x,y) \in U$. Therefore $\mathrm{supp}(a_{(x,y)}) \subseteq U[(x,y)]$. $\square$

Presently, we will see how the techniques of Bell and Nagórko can be applied to show that the finite coarse direct product preserves coarse property C [BN18].

Theorem I.47. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces with coarse property C. Then $(X \times Y, \mathcal{E} * \mathcal{F})$ has coarse property C.*

Proof. Let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence of entourages in $\mathcal{E} * \mathcal{F}$. For each i, put $K_i = (p_1 \times p_1)(E_i)$ and $L_i = (p_2 \times p_2)(E_i)$. Then, by the definition of $\mathcal{E} * \mathcal{F}$, each $K_i \in \mathcal{E}$ and $L_i \in \mathcal{F}$. Observe that since $E_i \subseteq E_{i+1}$, we have $K_i \subseteq K_{i+1}$ and $L_i \subseteq L_{i+1}$. Arrange the indices $1, 2, 3, \ldots$ into a two-dimensional array with the property that the indices are increasing from left to right along rows and from bottom to top along columns. In Figure I.3 we give one example of such an arrangement, which was first used in the metric proof [BN18]).

$$
\begin{array}{cccccc}
K_{10} & K_{14} & K_{19} & K_{25} & K_{32} & K_{40} \\
K_{6} & K_{9} & K_{13} & K_{18} & K_{24} & K_{31} \\
K_{3} & K_{5} & K_{8} & K_{12} & K_{17} & K_{23} \\
K_{1} & K_{2} & K_{4} & K_{7} & K_{11} & K_{16}
\end{array}
\qquad
\begin{array}{cccccc}
K_{1,4} & K_{2,4} & K_{3,4} & K_{4,4} & K_{5,4} & K_{6,4} \\
K_{1,3} & K_{2,3} & K_{3,3} & K_{4,3} & K_{5,3} & K_{6,3} \\
K_{1,2} & K_{2,2} & K_{3,2} & K_{4,2} & K_{5,2} & K_{6,2} \\
K_{1,1} & K_{2,1} & K_{3,1} & K_{4,1} & K_{5,1} & K_{6,1}
\end{array}
$$

Figure I.3. A Possible Rearrangement of the Sequence $K_1, K_2 \ldots$ Into a Two-dimensional Array $K_{i,j}$.

For each i, we apply the coarse property C definition to the column $K_{i,1}, K_{i,2}, \ldots$ to find an n_i and a cover $\mathcal{U}_{i,1}, \mathcal{U}_{i,2}, \ldots, \mathcal{U}_{i,n_i}$ of X by uniformly bounded families of subsets of X with the property that each $\mathcal{U}_{i,j}$ is $K_{i,j}$-disjoint. Then, consider the sequence $L_{1,n_1}, L_{2,n_2},, \ldots$. We may assume that the sequence is increasing by replacing L_{i,n_i} by an entourage that occurs higher in the i-th column, if necessary.

Using this sequence, and the fact that Y has coarse property C, we find a cover of Y by families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of subsets of Y that are uniformly bounded with the property that $\mathcal{V}_i$ is L_{i,n_i}-disjoint.

Using the same rearrangement of indices as above, we construct the doubly indexed collection $\{E_{i,j}\}$ of entourages from the given sequence $E_1 \subseteq E_2 \subseteq \cdots$. We claim that the family $\mathcal{W}_{i,j} = \{U \times V \colon U \in \mathcal{U}_{i,j}, V \in \mathcal{V}_i\}$ covers $X \times Y$, consists of uniformly bounded sets, and has the property that $\mathcal{W}_{i,j}$ is $E_{i,j}$-disjoint. To finish the proof, we simply need to unravel the re-indexing to arrive at the original sequence, which may now include some empty families.

First we check that the collection $\mathcal{W}_{i,j}$ covers $X \times Y$. To this end, let $(x, y) \in X \times Y$ be given. Since the families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ cover Y, there is a subset V in one such family (say) $\mathcal{V}_{i_0}$ such that $y \in V$ and $V \in \mathcal{V}_{i_0}$. Now, the families $\mathcal{U}_{i_0,1}, \mathcal{U}_{i_0,2}, \ldots, \mathcal{U}_{i_0,n_{i_0}}$ cover X, so there is some index (say) j_0 so that the family $\mathcal{U}_{i_0,j_0}$ contains a subset U of X with $x \in U$.

Fix a pair (i, j) and consider the family $\mathcal{W}_{i,j}$. To show that $\mathcal{W}_{i,j}$ is uniformly bounded, we must show that

$$\bigcup_{W \in \mathcal{W}_{i,j}} W \times W$$

is an entourage in the product coarse structure $\mathcal{E} * \mathcal{F}$. By the definition, we need only show the projection of this union to each factor is an entourage in that factor. Since each $W \in \mathcal{W}_{i,j}$ can be expressed as a product $W = U \times V$ with $U \in \mathcal{U}_{i,j}$ and $V \in \mathcal{V}_i$, we observe that,

$$\begin{aligned}
(p_1 \times p_1)\left[\bigcup_{W \in \mathcal{W}_{i,j}} W \times W\right] &= (p_1 \times p_1)\left[\bigcup_{W = U \times V} ((U \times V) \times (U \times V))\right] \\
&= \bigcup_{W = U \times V} [(p_1 \times p_1)((U \times V) \times (U \times V))] \\
&= \bigcup_{W = U \times V} [U \times U] \in \mathcal{E}.
\end{aligned}$$

The conclusion for the projection to the second factor is similar.

Finally, we check that $\mathcal{W}_{i,j}$ is $E_{i,j}$-disjoint. To this end, take distinct $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{W}_{i,j}$. Assume that there were some $(a, c, b, d) \in E_{i,j} \cap ((U_1 \times V_1) \times (U_2 \times V_2))$. Then, in particular, $a \in U_1$, $b \in U_2$, $c \in V_1$ and $d \in V_2$. Thus, $(a, b) \in (p_1 \times p_1)(E_{i,j}) = K_{i,j}$ and $(c, d) \in (p_2 \times p_2)(E_{i,j}) = L_{i,j}$. Since $U_1 \times V_1 \neq U_2 \times V_2$, we either have $U_1 \neq U_2$ or $V_1 \neq V_2$. In the first case, the $K_{i,j}$-disjointness of $\mathcal{U}_{i,j}$ does not allow (a, b) to be in $K_{i,j}$. In the second case, the fact that $\mathcal{V}_i$ is L_{i,n_i}-disjoint and the fact that

$L_{i,j} \subseteq L_{i,n_i}$ for all $j \leq n_i$ means that (c, d) cannot be in $L_{i,j}$. Thus, there can be no such point (a, c, b, d). We conclude that the intersection is empty, which is what we needed to show.

$\square$

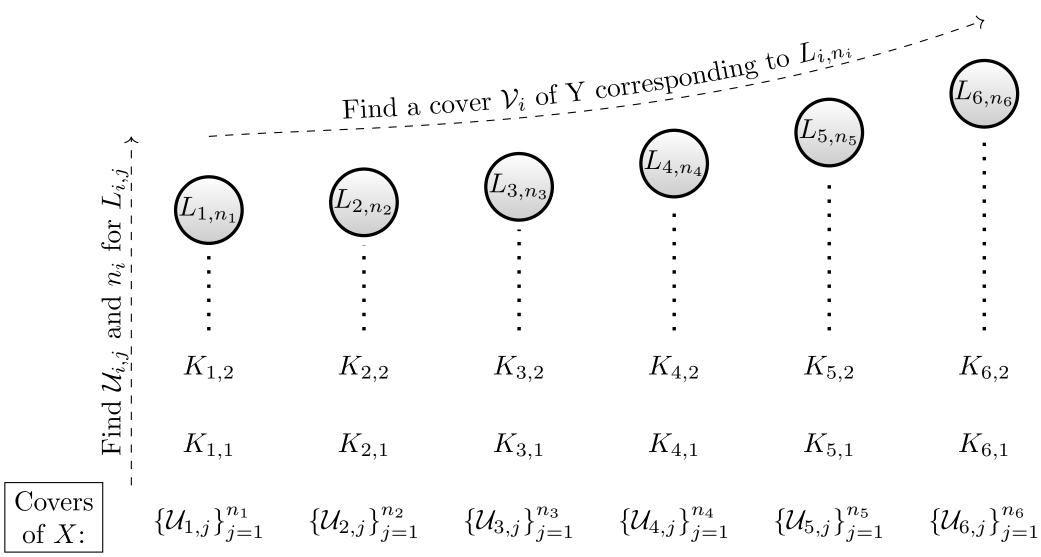


Figure I.4. We Find Covers $\{\mathcal{U}_{i,j}\}_{j=1}^{n_i}$ of X for Each Column $K_{i,1}, K_{i,2}, \ldots,$ and Then Construct a Cover $\{\mathcal{V}_i\}_{i=1}^{m}$ of Y Corresponding to L_{i,n_i}

I.4. Coarse Countable Asymptotic Dimension

This final section describes our generalization of Dydak's countable asymptotic dimension to the coarse category. We will show that the coarse analog of countable asymptotic dimension is equivalent to straight coarse finite decomposition complexity and therefore that it is preserved by the coarse direct product.

We begin by recalling Dydak's original definition [Dyd16]: The metric space X is said to have **countable asymptotic dimension** if there is a sequence of positive integers (n_i) $(i = 1, 2, \ldots)$ so that for any sequence of positive real numbers (R_i), there is a sequence of families $\mathcal{V}_i$ of subsets of X such that

1. $\mathcal{V}_1 = \{X\}$;

2. each $U \in \mathcal{V}_i$ can be expressed as a union of at most n_i families from $\mathcal{V}_{i+1}$ that are R_i-disjoint; and

3. at least one of the families $\mathcal{V}_i$ is uniformly bounded.

Definition I.48. The coarse space $(X, \mathcal{E})$ is said to have **countable coarse dimension** (CCD) if there is a sequence n_i of positive integers so that for any sequence of entourages L_i there is a sequence of families $\mathcal{V}_i$ so that

CCD1 $\mathcal{V}_1 = \{X\}$;

CCD2 for every i, each $U \in \mathcal{V}_i$ admits an (L_i, n)-decomposition over $\mathcal{V}_{i+1}$ with $n \leq n_i$; and

CCD3 at least one collection $\mathcal{V}_i$ is uniformly bounded.

It is clear from the definition that CCD passes to subsets.

Proposition I.49. *Let $(X, \mathcal{E})$ have CCD, suppose that $Y \subseteq X$, and consider the coarse structure $\mathcal{E}|_Y$ obtained from $\mathcal{E}$ by restricting elements $E \in \mathcal{E}$ to $Y \times Y$. Then $(Y, \mathcal{E}|_Y)$ has CCD.*

It is straightforward to show that CCD is a coarse invariant. Alternatively, this can be concluded from Proposition I.53 and the corresponding result for straight finite coarse decomposition complexity [BMN17, Proposition 3.3].

Proposition I.50. *Let $f : X \to Y$ be a coarsely uniform embedding. If Y has CCD, then X has CCD.*

Proof. By Proposition I.49, we may assume that f is onto to ease notation. Let (n_i) be the sequence of positive integers from the CCD condition on Y. Given a sequence (K_i) of elements of $\mathcal{E}$ we put $L_i = (f \times f)(K_i)$ and observe that since f is coarse, this is a collection of sets in $\mathcal{F}$.

Using the fact that Y has CCD, take $(\mathcal{V}_i)$ with the property that $\mathcal{V}_1 = \{Y\}$, each $U \in \mathcal{V}_i$ admits an (K_i, n_i)-decomposition over $\mathcal{V}_{i+1}$ and with the property that some $\mathcal{V}_{i_0}$ is uniformly bounded. For each i, put $\mathcal{U}_i = \{f^{-1}(V) \colon V \in \mathcal{V}_i\}$. Property CCD1 is easy to see for $\mathcal{U}_1$.

To see CCD2, we let $U \in \mathcal{U}_i$. Then, with $V = f(U)$ we see that V admits a (L_i, n_i)-decomposition over $\mathcal{V}_{i+1}$. Thus, it remains only to show that the inverse image under f of a L_i-disjoint family in Y is K_i disjoint in X. To this end, let $A \neq B$ have the property that $(A \times B) \cap L_i = \emptyset$. Then, with $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$, we find that $A' \times B' \cap K_i \subseteq (f \times f)^{-1}((A \times B) \cap L_i) = \emptyset$.

Finally, for CCD3, we denote $\bigcup_{V \in \mathcal{V}_{i_0}} V \times V$ by $\Delta_{\mathcal{V}_{i_0}}$. Then, we have that $\Delta_{\mathcal{U}_{i_0}} = \bigcup_{V \in \mathcal{V}_{i_0}} f^{-1}(V) \times f^{-1}(V) = (f \times f)^{-1}(\bigcup_{V \in \mathcal{V}_{i_0}} V \times V) = (f \times f)^{-1}(\Delta_{\mathcal{V}_{i_0}})$. By assumption $\mathcal{V}_{i_0}$ is uniformly bounded and so $\Delta_{\mathcal{V}_{i_0}}$ is an entourage. Since f is a coarsely uniform embedding, we have that $(f \times f)^{-1}(\Delta_{\mathcal{V}_{i_0}}) = \Delta_{\mathcal{U}_{i_0}}$ is an entourage and therefore $\mathcal{U}_{i_0}$ is uniformly bounded. $\square$

It is straightforward to verify that our definition of CCD reduces to countable asymptotic dimension in the case of a metric space in the bounded coarse structure. We omit the proof and refer to the technique used in for sFCDC [BMN17, Proposition 3.4].

Proposition I.51. *Let X be a metric space and let $\mathcal{E}$ be the bounded coarse structure on X. Then X has countable asymptotic dimension if and only if X has coarse countable dimension.*

To prove the main theorem of this section, we need a simple lemma that is a straightforward analog of a statement due to Dydak and Virk [DV16, Corollary 8.3]; we note that their proof does not use the metric formulation of countable asymptotic dimension in any meaningful way. We omit the proof, which follows the same scheme as the original statement.

Lemma I.52. *In the definition of CCD, we may assume each $\mathcal{V}_i$ to be a partition of X.*

We are now in a position to prove the generalization of Theorem 8.4 of Dydak and Virk [DV16] to the coarse category.

Proposition I.53. *Let $(X, \mathcal{E})$ be a coarse space. The following are equivalent:*

1. *there is a sequence (n_i) of integers such that for every sequence of entourages K_i there is a finite sequence of families $\mathcal{V}_1, \ldots, \mathcal{V}_r$ of subsets of X such that $\mathcal{V}_1 = \{X\}$, every $V \in \mathcal{V}_i$ admits an (K_i, n_i)-decomposition over $\mathcal{V}_{i+1}$ and such that $\mathcal{V}_r$ is uniformly bounded.*

2. *for every sequence L_i of entourages there is a finite sequence of families $\mathcal{U}_1, \ldots, \mathcal{U}_s$ of subsets of X such that $\mathcal{U}_1 = \{X\}$, every $U \in \mathcal{U}_i$ admits a $(L_i, 2)$-decomposition over $\mathcal{U}_{i+1}$ and such that $\mathcal{U}_s$ is uniformly bounded.*

Proof. Clearly (2) implies (1).

To see the other implication, let (n_i) be the sequence of positive integers satisfying (1) for X. Let $L_1, L_2, \ldots$ be a sequence of entourages. By taking unions we may assume $L_i \subseteq L_{i+1}$. Put $K_1 = L_{n_1}$, $K_2 = L_{n_1+n_2}$, and in general, put $K_j = L_{n_1+\cdots+n_j}$. Apply (1) with the sequence (K_i) to obtain $\mathcal{V}_1, \mathcal{V}_2, \ldots$ such that $\mathcal{V}_1 = \{X\}$ and such that $\mathcal{V}_i$ admits a (K_i, n_i)-decomposition over $\mathcal{V}_{i+1}$.

We will define a sequence $\mathcal{U}_i$ of families of subsets of X with the property that $\mathcal{U}_1 = \{X\}$ and $\mathcal{U}_i$ admits an $(L_i, 2)$-decomposition over $\mathcal{U}_{i+1}$. To begin, we observe that we can write $X = X^1 \cup X^2 \cup \cdots \cup X^{n_1}$, with each $X^i = \bigsqcup_{K_1-\text{disj}} X^i_j$, and each $X^i_j \in \mathcal{V}_2$. Therefore, we take $\mathcal{U}_2 = \{X^1_1, X^1_2, X^1_3, \ldots\} \cup \{X^2 \cup X^3 \cup \cdots \cup X^{n_1}\}$. Then, it is clear that X can be $(K_1, 2)$-decomposed over $\mathcal{U}_2$ and since $L_1 \subseteq L_{n_1} = K_1$, we see that there is an $(L_1, 2)$-decomposition of any set in $\mathcal{U}_1$ over the family $\mathcal{U}_2$. For $\mathcal{U}_3$, we take $\{X^1_1, X^1_2, \ldots\} \cup \{X^2_1, X^2_2, \ldots\} \cup \{X^3 \cup X^4 \cup \cdots \cup X^{n_1}\}$; we also observe that any set in $\mathcal{U}_2$ admits an $(L_2, 2)$-decomposition over $\mathcal{U}_3$ since $L_2 \subseteq L_{n_1} = K_1$. Continue to define families this way to obtain $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n_1}$ with an $(L_i, 2)$-decomposition of each $\mathcal{U}_i$ over $\mathcal{U}_{i+1}$ for $1 \leq i < n_1$. We observe that in this way, $\mathcal{U}_{n_1} = \mathcal{V}_2$.

To define $\mathcal{U}_j$ for $j \geq n_1 + 1$, we observe first that we may assume each family $\mathcal{V}_i$ is a partition of X itself, (cf. [DV16, Corollary 8.3]). We then repeat the above procedure to arrive at $\mathcal{U}_{n_1+n_2} = \mathcal{V}_3$. We repeat this entire process until we arrive at $\mathcal{U}_{n_1+\cdots+n_{r-1}} = \mathcal{V}_r$, which is uniformly bounded. □

A coarse space has sFCDC precisely when it satisfies condition (2) of Proposition I.53. Condition (1) of Proposition I.53 is the coarse analog of countable asymptotic dimension. Since sFCDC is preserved by coarse direct products [BMN17, Theorem 4.17], we obtain the following.

Corollary I.54. *The coarse version of Dydak's countable asymptotic dimension is preserved by coarse direct products.* □

Yamauchi recently showed that coarse straight finite decomposition complexity implies coarse property A [Yam17, Theorem 3.2]. Combining that result with Proposition I.53 we immediately obtain the following result.

Theorem I.55. *Let $(X, \mathcal{E})$ be a space with coarse countable dimension; then $(X, \mathcal{E})$ has coarse property A.* $\square$

Bell, Moran, and Nagórko showed that sFCDC is preserved by coarse products [BMN17, Theorem 4.17]. Thus, we immediately obtain the following result.

Theorem I.56. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces with countable coarse dimension. Then their coarse direct product has coarse countable dimension.* $\square$

CHAPTER II

FREE PRODUCTS

II.1. Total Spaces

Definition II.1. Let $(X, \mathcal{E})$ be a coarse space. Let $\mathcal{U} = \{U_i\}_{i \in J}$ be a collection of subsets of X. We define a coarse space called the **total space** [Gue14] $T(\mathcal{U}; J)$ of the $\{U_i\}_{i\in J}$ as follows. The underlying set is the disjoint union, $\sqcup_{i\in J} U_i$. The entourages are the disjoint unions

$$\sqcup_{i\in J}\big(E \cap (U_i \times U_i)\big) \subseteq \sqcup_{i\in J}(U_i \times U_i),$$

where E ranges over $\mathcal{E}$.

Definition II.2. Let X and Y be coarse spaces. A property $\mathscr{P}$ of coarse spaces is **coarse invariant** if whenever X has $\mathscr{P}$ and X is coarsely equivalent to Y, then Y has $\mathscr{P}$. In this case, $\mathscr{P}$ is called a **coarse property**.

Definition II.3. Let $\mathscr{P}$ be a coarse property. Following Guentner [Gue14], we say that the family $\{U_i\}_{i\in J}$ has property $\mathscr{P}$ **uniformly** if the total space $T(\mathcal{U}; J)$ has $\mathscr{P}$.

Definition II.4. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces. Let $f : X \to Y$ be a map and $F \in \mathcal{F}$. A **coarse fiber** of f at scale $F \in \mathcal{F}$ is any set $A \subseteq X$ that satisfies $(f \times f)(A \times A) \subseteq F$; i.e., for any $x, y \in A$ we have $(f(x), f(y)) \in F$.

Definition II.5. Let $n \in \mathbb{N}$. We say that **coarse fibers of f have asymptotic dimension of at most n uniformly** if for every $L \in \mathcal{E}$ and $F \in \mathcal{F}$ there is some $K \in \mathcal{E}$ so that whenever A is a coarse fiber of f at scale F, there exist families of subsets, $\mathcal{U}_0, \mathcal{U}_1, \ldots \mathcal{U}_n$, of A that are K-bounded, L-disjoint, and $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ covers A.

The following proposition shows the Definition II.5 version of uniform asymptotic dimension coincides with the total space definition of uniform asymptotic dimension.

Proposition II.6. *Let $\mathcal{U} = \{U_i\}_{i \in J}$ be a collection of subsets of $(X, \mathcal{E})$. Then $T(\mathcal{U}; J)$ has asymptotic dimension at most k if and only if for every $E \in \mathcal{E}$ there is a $K \in \mathcal{E}$ and families ${\mathcal{V}_0}^i, {\mathcal{V}_1}^i, \ldots, {\mathcal{V}_k}^i$ so that for each i, $\cup_j {\mathcal{V}_j}^i$ covers U_i, each ${\mathcal{V}_j}^i$ is E-disjoint, and each ${\mathcal{V}_j}^i$ is K-bounded.*

Proof. We define maps $g_i : U_i \to T(\mathcal{U}; J)$ by $x \mapsto (x; i)$. Suppose first that $\operatorname{asdim} T(\mathcal{U}; J) \leq k$. Let $E \in \mathcal{E}$ be given. Put $E_i = E \cap U_i \times U_i$. Then, $\sqcup E_i$ is an entourage in the total space, so we can find $\mathcal{V}_0, \ldots, \mathcal{V}_k$ such that $\cup \mathcal{V}_j$ covers $T(\mathcal{U}; J)$, $\mathcal{V}_j$ are uniformly bounded, and each $\mathcal{V}_j$ is $\sqcup E_i$-disjoint. Since each of the $\mathcal{V}_j$ are uniformly bounded, there is some $K \in \mathcal{E}$ so that $\bigcup_{j=0}^{k} \left(\bigcup_{V \in \mathcal{V}_j} V \times V \right) = \bigsqcup_i K \cap (U_i \times U_i)$. Put ${\mathcal{V}_j}^i = \{g_i^{-1}(V) \colon V \in \mathcal{V}_j\}$ for $j = 0, \ldots k$. Clearly then the resulting families ${\mathcal{V}_0}^i, \ldots {\mathcal{V}_k}^i$ are K-bounded and cover U_i for each i. Moreover, $V \neq V' \in {\mathcal{V}_j}^i$ implies $\emptyset = (V \times V') \cap E_i = (V \times V') \cap (U_i \times U_i) \cap E = (V \times V') \cap E$ as desired.

On the other hand, suppose the second condition holds for $\mathcal{U}$. Let L be an entourage of the total space $T(\mathcal{U}; J)$. Then there is some $E \in \mathcal{E}$ so that $L = \sqcup_{i \in J}(E \cap (U_i \times U_i))$. By assumption, we can find a $K \in \mathcal{E}$ so that for each i there are families ${\mathcal{V}_0}^i, \ldots {\mathcal{V}_k}^i$ that cover U_i and are E-disjoint, K-bounded. Furthermore, we may assume these are families of subsets of U_i. Define $\mathcal{V}_j = \sqcup_{i \in J} g_i({\mathcal{V}_j}^i)$. It is clear the resulting sequence of families covers $T(\mathcal{U}; J)$ and since each ${\mathcal{V}_j}^i$ is E-disjoint, we see that $\mathcal{V}_j$ is L-disjoint. Note that for each j and for each i, $\cup_{V\in\mathcal{V}_j{}^i} V\times V\subseteq K$ implies $\sqcup_{i\in J}\cup_{V\in\mathcal{V}_j{}^i} V\times V\subseteq\sqcup_{i\in J}K\cap(U_i\times U_i)$. $\square$

II.2. The Coarse Free Product

Let $(X,\mathcal{E})$ be a coarse space. Fix a basepoint $x_0\in X$. Consider the collection $*X$ of words in the alphabet $X\setminus\{x_0\}$ along with the empty word $\boldsymbol{\varepsilon}$. We define the concatenation $\mathbf{x}\cdot\mathbf{x}'$ of two words in the usual way, so that $\mathbf{x}\cdot\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}\cdot\mathbf{x}=\mathbf{x}$. For each distinct pair of elements $\mathbf{x}$ and $\mathbf{x}'\in *X$ there is a unique $\mathbf{a}\in *X$ with the properties that $\mathbf{x}=\mathbf{a}b\mathbf{c}$ and $\mathbf{x}'=\mathbf{a}b'\mathbf{c}'$ with $b\neq b'$, b,b' both in $X\setminus\{x_0\}$, and $\mathbf{c},\mathbf{c}'$ both in $*X$. (Note that we allow the words $\mathbf{a},\mathbf{c}$ or $\mathbf{c}'$ to be $\boldsymbol{\varepsilon}$.) When necessary, we use the notation $\mathbf{a}=\mathbf{x}\wedge\mathbf{x}'$ for this element.

Let $E\in\mathcal{E}$ be given. Define $\|\boldsymbol{\varepsilon}\|_E=\|\boldsymbol{\varepsilon}\|^E=0$; for nonempty $\mathbf{x}=x_1x_2\cdots x_k\in *X$ define $\|\mathbf{x}\|_E=\sum_{i=1}^k D_E(x_0,x_i)$ and $\|\mathbf{x}\|^E=\sum_{i=1}^k D_E(x_i,x_0)$. In the case that E is symmetric, $\|\mathbf{x}\|_E=\|\mathbf{x}\|^E$. Moreover define $D_E^*: *X\times *X\to\mathbb{Z}\cup\{\infty\}$ by

$$D_E^*(\mathbf{x},\mathbf{x}')=\begin{cases}0 & \mathbf{x}=\mathbf{x}'\\ D_E(b,b')+\|\mathbf{c}\|_E+\|\mathbf{c}'\|^E & \mathbf{x}\neq\mathbf{x}'.\end{cases}$$

Definition II.7. Let $(X,\mathcal{E})$ be a coarse space and $x\in X$ be fixed. Let $E\in\mathcal{E}$ We define the **symmetric ball** of size E about x to be $B_E(x)=E_x\cup E^x$ where $E_x=\{y\in X\mid (x,y)\in E\}$ and $E^x=\{y\in X\colon (y,x)\in E\}$

Proposition II.8. *If $E\in\mathcal{E}$ is symmetric, then D_E^* is an ∞-metric on $*X$.*

Proposition II.9. *Let $n\in\mathbb{N}$ and put $\langle E,n\rangle=\{(\mathbf{x},\mathbf{x}')\in *X\times *X\colon D_E^*(\mathbf{x},\mathbf{x}')\leq n\}$. Define the collection $*\mathcal{E}$ to be the subset closure of $\{\langle E,n\rangle\colon E\in\mathcal{E},n\in\mathbb{N}\}$. Then, $*\mathcal{E}$ is a coarse structure on $*X$.*

Proof. We must show that $*\mathcal{E}$ (a) contains the diagonal, (b) is closed under inverses, (c) is closed finite under unions, (d) is closed under subsets, and (e) is closed under compositions.

(a) It is clear that $\langle \Delta, 0\rangle$ contains the diagonal in $*X \times *X$.

(b) Given $L \in *\mathcal{E}$, we take a symmetric E so that $\langle E, n\rangle$ contains L. It is easy to see that $\langle E^{-1}, n\rangle = \langle E, n\rangle^{-1}$.

(c) Let L and L' be given elements of $*\mathcal{E}$. Find $E, E' \in \mathcal{E}$ and $n, n' \in \mathbb{N}$ so that $L \subseteq \langle E, n\rangle$ and $L' \subseteq \langle E', n'\rangle$. Then $L \cup L' \subseteq \langle E, n\rangle \cup \langle E', n'\rangle \subseteq \langle E \cup E', n + n'\rangle$.

(d) Holds by definition.

(e) Let L and L' be given elements of $*\mathcal{E}$. Find $E, E' \in \mathcal{E}$ and $n, n' \in \mathbb{N}$ so that $L \subseteq \langle E, n\rangle$ and $L' \subseteq \langle E', n'\rangle$. Then, $\langle E, n\rangle \circ \langle E', n'\rangle \subseteq \langle E \cup E', n + n'\rangle$.

□

Definition II.10. Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0 \in X$. The coarse space $(*X, *\mathcal{E})$ constructed above is called the **coarse free product of** $(X, \mathcal{E})$.

We define the free product $X * Y$ of pointed spaces (X, x_0) and (Y, y_0) to be $*(X \vee Y)$.

Definition II.11. Let $(X, \mathcal{E})$ be a coarse space and $x_0 \in X$ be fixed. Suppose $\mathbf{w} = x_1 x_2 \cdots x_n \in *X, x_i \neq x_0$. We say the **order (or length)** of $\mathbf{w}$ is n and write $\operatorname{ord}(\mathbf{w}) = n$. Note that $\operatorname{ord} \mathbf{w} = 0$ if and only if $\mathbf{w} = \boldsymbol{\varepsilon}$.

Suppose (X, d) is a discrete metric space with fixed point x_0. The (metric) free product $*X$ was defined by Bell and Nagórko [BN18].

They showed that the function $d^* : *X \times *X \to [0, \infty)$ is a metric, where d^* is defined by $d^*(\mathbf{x}, \mathbf{x}) = 0$ and if $\mathbf{x} \neq \mathbf{x}'$ are expressed as $\mathbf{x} = \mathbf{a}bx_1 \cdots x_n$, $\mathbf{x}' = \mathbf{a}b'x_1' \cdots x_m'$, then

$$d^*(\mathbf{a}bx_1 \ldots x_n, \mathbf{a}b'x_1' \ldots x_m') = d(b, b') + \sum_{i=1}^{n} d(x_i, x_0) + \sum_{j=1}^{m} d(x_j, x_0).$$

Proposition II.12. *Let $\mathcal{E}$ be the bounded coarse structure on the metric space (X, d) inherited from d, let $*\mathcal{E}$ be the resulting free product coarse structure, and let $\mathcal{F}$ be the bounded coarse structure on $*X$ inherited from d^*. Then $\mathcal{F} = *\mathcal{E}$.*

Proof. Since X is a discrete metric space, let $R = \inf\{d(y, y') \mid y \neq y' \in X\}$. We observe that $R > 0$. Let $L \in *\mathcal{E}$. We may assume L is of the form $\langle E, n \rangle$ where $E = \{(y, y') \in X \mid d(y, y') \leq m\}$ for some $m, n \in \mathbb{N}$. Suppose $(\mathbf{x}, \mathbf{x}') \in L$. Then

$$D_E^*(\mathbf{x}, \mathbf{x}') = D_E^*(\mathbf{a}bx_1 \ldots x_p, \mathbf{a}b'x_1 \ldots x_q) \leq n$$

But this means that $d^*(\mathbf{x}, \mathbf{x}') \leq mn$ thus $L \in \mathcal{F}$.

On the other hand, if $L \in \mathcal{F}$ we may assume $L = \{(\mathbf{x}, \mathbf{x}') \in *X \times *X \mid d^*(\mathbf{x}, \mathbf{x}') \leq m \cdot R\}$ for some $m \in \mathbb{N}$. This means if $(\mathbf{x}, \mathbf{x}') = (\mathbf{a}b\mathbf{c}, \mathbf{a}b'\mathbf{c}') \in L$ then $\operatorname{ord}(b\mathbf{c}) \leq m$ and $\operatorname{ord}(b\mathbf{c}) \leq m$. Thus $L \subseteq \langle E, m \rangle \in *\mathcal{E}$ and we are done. □

II.3. Free-Product Permanence and Fibering

In this section, we provide an approach for dealing with preservation of coarse properties by coarse free products. This approach is similar to the one given by Guentner [Gue14] regarding metric spaces. We consider a property $\mathscr{P}$ that is a coarse invariant, e.g. finite asymptotic dimension. We prove that the property $\mathscr{P}$ is preserved by the coarse free product construction whenever trees have $\mathscr{P}$ and $\mathscr{P}$ satisfies union permanence and fibering permanence, which are described in further detail below.

Definition II.13. A coarse property $\mathscr{P}$ is said to satisfy **excisive union permanence** if X has $\mathscr{P}$ whenever the following conditions are satisfied:

(1) $X = \cup X_\alpha$ where $\{X_\alpha\}$ has uniform $\mathscr{P}$; and

(2) for every $E \in \mathcal{E}$ there is a $Y_E \subseteq X$ with $\mathscr{P}$ so that $\{X_\alpha \setminus Y_E\}$ is E-disjoint.

Definition II.14. A coarse property $\mathscr{P}$ is said to satisfy **fibering permanence** if, X has $\mathscr{P}$ whenever $f : X \to Y$ is a uniformly expansive map of the coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$, Y has $\mathscr{P}$, and for each $F \in \mathcal{F}$ the coarse fibers of f at scale F have $\mathscr{P}$ uniformly.

Theorem II.15. *Let $\mathscr{P}$ be a coarse property of coarse spaces that satisfies excisive union permanence and fibering permanence. Suppose trees have $\mathscr{P}$. Then, the coarse space $(*X, *\mathcal{E})$ has property $\mathscr{P}$ whenever $(X, \mathcal{E})$ does.*

Lemma II.16. *Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0 \in X$ and consider the free product $*X$. Let $\mathscr{P}$ be a coarse invariant. Let $A \subseteq *X$ have $\mathscr{P}$. Then, for any subset $Y \subseteq *X$, the family $\{\mathbf{y} \cdot A\}_{\mathbf{y} \in Y}$ has property $\mathscr{P}$ uniformly.*

Proof. We form the total space $T(A; Y)$ of the family $\{A\}_{\mathbf{y} \in Y}$ (indexed by $\mathbf{y} \in Y$) and observe that this total space has property $\mathscr{P}$.

We define a map $f : T(A; Y) \to T(\mathbf{y} \cdot A; Y)$ by $(\mathbf{a}, \mathbf{y}) \mapsto (\mathbf{y} \cdot \mathbf{a}, \mathbf{y})$. It remains to show that this is a coarse equivalence. If E is an entourage in $T(A; Y)$, then $E \cap [(A, \mathbf{y}) \times (A, \mathbf{y})]$ is an entourage in $*\mathcal{E}$. Therefore, there is some $K \in \mathcal{E}$ and an $n \in \mathbb{N}$, for which $E \subseteq \langle K, n \rangle$. Then, $((\mathbf{a}, \mathbf{y}), (\mathbf{a}', \mathbf{y})) \in E$ implies $D^*_K(\mathbf{a}, \mathbf{a}') \leq n$. But, $D^*_K(\mathbf{a}, \mathbf{a}') = D^*_K(\mathbf{y} \cdot \mathbf{a}, \mathbf{y} \cdot \mathbf{a}')$, so that $((\mathbf{y} \cdot \mathbf{a}, \mathbf{y}), (\mathbf{y} \cdot \mathbf{a}', \mathbf{y})) \in \langle K, n \rangle \cap [(\mathbf{y}A, \mathbf{y}) \times (\mathbf{y}A, \mathbf{y})]$, which is an entourage in $T(\mathbf{y} \cdot A, \mathbf{y})$. The opposite inclusion follows the same scheme and so we omit it. □

Lemma II.17. *Let $\mathscr{P}$ be a coarse property of coarse spaces that satisfies excisive union permanence. Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0 \in X$. Put $X^{(n)} = \{x \in *X \colon \operatorname{ord}(x) = n\}$. If X has property $\mathscr{P}$, then for each n, $X^{(n)}$ has property $\mathscr{P}$.*

Proof. We apply induction. Put $X^* = X \setminus \{x_0\}$. Observe that $X^{(1)} = X^*$. Suppose the conclusion holds for $X^{(n-1)}$. Observe that $X^{(n)} = \bigcup_{\mathbf{x} \in X^{(n-1)}} \mathbf{x} \cdot X^*$. For each fixed $\mathbf{x} \in X^{(n-1)}$, the set $\mathbf{x} \cdot X^*$ is coarsely equivalent to X^*. Thus, $\mathbf{x} \cdot X^*$ has $\mathscr{P}$ for each $\mathbf{x} \in X^{(n-1)}$. By Lemma II.16, the collection $\{\mathbf{x} \cdot X^* \mid \mathbf{x} \in X^{(n-1)}\}$ has $\mathscr{P}$ uniformly.

Let $L \in *\mathcal{E}$ be given. Then, we can find a symmetric entourage $E \in \mathcal{E}$ and a natural number m so that $L \subseteq \langle E, m \rangle$. Put $Y_{\langle E,m \rangle} = X^{(n-1)} \cdot (B_{E^m}(x_0) \cap X^*)$. A routine check shows $Y_{\langle E,m \rangle}$ is coarsely equivalent to $X^{(n-1)}$. Thus we see that $Y_{\langle E,m \rangle}$ has property $\mathscr{P}$ by the inductive hypothesis. Finally, we show that the collection $\{\mathbf{x} \cdot X^* \setminus Y_{\langle E,m \rangle} \colon x \in X^{(n-1)}\}$ is $\langle E, m \rangle$-disjoint. To this end if $\mathbf{w} \neq \mathbf{w}'$ are in this collection then we may write $\mathbf{w} = \mathbf{a} \cdot b \cdot \mathbf{c} \cdot x$ and $\mathbf{w}' = \mathbf{a} \cdot b' \cdot \mathbf{c}' \cdot x'$ where $\mathbf{a}, \mathbf{c}, \mathbf{c}' \in *X$, $b, b', x, x' \in X$, $b \neq b'$ and $x, x' \notin E^m$. Then

$$D_E^*(\mathbf{w}, \mathbf{w}') = D_E(b, b') + \|\mathbf{c}\|_E + \|\mathbf{c}'\|^E + D_E(x_0, x) + D_E(x', x_0) \geq m$$

by assumption since $x, x' \notin E^m$. Therefore the collection is $\langle E, m \rangle$-disjoint (hence L-disjoint).

We apply excisive union permanence to complete the proof. $\square$

With these lemmas, we now prove Theorem II.15

Proof. Let T be a graph whose vertex set is in one-to-one correspondence with the elements of $*X$. We denote by $t_{\mathbf{x}}$ the vertex of T corresponding to the element $\mathbf{x} \in *X$. We connect two vertices $t_{\mathbf{x}_1}$ and $t_{\mathbf{x}_2}$ of T by an edge if and only if there is an $x \in X$

$(x \neq x_0)$ for which $\mathbf{x}_1 x = \mathbf{x}_2$ or $\mathbf{x}_2 x = \mathbf{x}_1$ (as elements of $*X$). It is clear that T is a tree. Give T the bounded coarse structure it inherits as a metric space.

Define $f : *X \to T$ by $f(\mathbf{x}) = t_{\mathbf{x}}$. We claim that f is uniformly expansive. To this end, let $L \in *\mathcal{E}$ be given. Then, we can find an $E \in \mathcal{E}$ and an $n \in \mathbb{N}$ so that $L \subseteq \langle E, n\rangle$. Suppose $\mathbf{x} \neq \mathbf{x}'$ and that $(\mathbf{x}, \mathbf{x}') \in L$. Put $\mathbf{a} = \mathbf{x} \wedge \mathbf{x}'$, find $b \neq b'$ in X and sequences $x_1, x_2, \ldots x_m$ and $x'_1, x'_2, \ldots, x'_{m'}$ of elements of $X \setminus \{x_0\}$ so that $\mathbf{x} = \mathbf{a} b x_1 \cdots x_m$ and $\mathbf{x}' = \mathbf{a} b' x'_1 \cdots x'_{m'}$.

Then,

$$n \geq D^*_E(\mathbf{x}, \mathbf{x}') = D_E(b, b') + \sum_{i=1}^{m} \|x_i\|_E + \sum_{i=1}^{m'} \|x'_i\|^E \geq 1 + m + m' = d_T(t_{\mathbf{x}}, t_{\mathbf{x}'}) - 1.$$

Thus, for all pairs $(\mathbf{x}, \mathbf{x}') \in L$, we have $d_T(t_{\mathbf{x}}, t_{\mathbf{x}'}) \leq n + 1$. Therefore, the image $(f \times f)(L)$ is a uniformly bounded set, which means f is uniformly expansive.

Let F be an entourage in the bounded coarse structure on T. Then, there is a $t = t_{\mathbf{y}} \in T$ (with $\mathbf{y} \in *X$) and an $R > 0$ so that $F \subseteq B_R(t) \times B_R(t) \subseteq T \times T$. Then, the set $\{t_{\mathbf{z}} \colon \mathbf{z} = \mathbf{y} \cdot x_1 \cdots x_k, x_i \in X \setminus \{x_0\}, k \leq 2R\}$ contains $B_R(t)$. We observe that if $A \subseteq *X$ is a coarse fiber of f at scale F, then $A \subseteq \mathbf{y} \cdot X^{(\leq 2R)}$.

By Lemma II.16 and Lemma II.17, coarse fibers of f have $\mathscr{P}$ uniformly. Since $\mathscr{P}$ is assumed to satisfy fibering permanence, we are done. $\square$

We note that any tree (in the bounded coarse structure) has asymptotic dimension 1 [Gro93] and therefore has coarse property A, coarse property C, as well as finite weak coarse decomposition complexity, finite coarse decomposition complexity, and straight finite decomposition complexity [BMN17].

Guentner shows that coarse property A satisfies fibering and excisive union permanence [Gue14, Theorem 6.5, Theorem 6.3]. Therefore, Theorem II.15 implies:

Corollary II.18. *Let $(X, \mathcal{E})$ be a coarse space with coarse property A. Let x_0 be a basepoint. Then, the coarse free product $*X$ has coarse property A.*

Bell, Moran, and Nagórko show that $\mathscr{P}$ satisfies fibering permanence when $\mathscr{P}$ is finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity [BMN17, Theorem 4.14]. Moreover, each of these properties satisfy excisive union permanence [BMN17, Theorem 4.18].

Theorem II.19. *Let $\mathscr{P}$ be one of the coarse properties: finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity. Then, $\mathcal{P}$ satisfies fibering permanence and excisive union permanence.*

Therefore, Theorem II.15 immediately implies:

Corollary II.20. *Let $(X, \mathcal{E})$ be a coarse space with a property $\mathscr{P}$ among finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity. Let x_0 be a basepoint. Then, the coarse free product $*X$ has $\mathscr{P}$.*

It is not known whether coarse property C satisfies fibering permanence, so we cannot use Theorem II.15 to show that coarse free products preserve coarse property C. We prove this using techniques similar to Bell and Nagórko [BN18] in Section II.5.

II.4. Asymptotic Dimension of a Free Product

By applying permanence results, we can show that finite asymptotic dimension is preserved by taking coarse free products as above. Instead, we apply the techniques of Theorem II.15 to find an upper bound for the asymptotic dimension of a coarse free product.

The asymptotic dimension of a metric space was defined by Gromov [Gro93]. Later, Grave [Gra05] and Roe [Roe03] provided definitions of asymptotic dimension of coarse spaces as follows.

We need the following union permanence result for coarse asymptotic dimension.

Theorem II.21. *[BMN17, Theorem 3.17] Suppose that $X = \bigcup_\alpha X_\alpha$, where $\operatorname{asdim} X_\alpha \leq n$ uniformly and for each entourage $L \in \mathcal{E}$ there is a subset $Y_L \subseteq X$ with $\operatorname{asdim} Y_L \leq n$ such that $\{X_\alpha \setminus Y_L\}$ forms an L-disjoint collection. Then, $\operatorname{asdim} X \leq n$.*

Theorem II.21 immediately implies the following version of Lemma II.17 for asymptotic dimension:

Lemma II.22. *Let $(X, \mathcal{E})$ be a coarse space with $\operatorname{asdim}(X) \leq k$ and fix $x_0 \in X$. As above, define $X^{(n)} = \{\mathbf{x} \in *X \colon \operatorname{ord}(x) = n\}$. Then $\operatorname{asdim}(X^{(n)}) \leq k$.*

Lemma II.23. *If $f : X \to Y$ is a uniformly expansive map of coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ with $\operatorname{asdim} Y \leq k$ and if coarse fibers of f have asymptotic dimension n uniformly for some $n \in \mathbb{N}$ then $\operatorname{asdim} X \leq (n+1)(k+1) - 1$*

Proof. Let $L \in \mathcal{E}$ be given. Since $\operatorname{asdim} Y \leq k$, we can find $k+1$-many $(f \times f)(L)$-disjoint families $\mathcal{V}_0, \ldots, \mathcal{V}_k$ of uniformly bounded subsets of Y.

Next, for each $V \in \cup_i \mathcal{V}_i$, since coarse fibers of f have asdim $\leq n$ uniformly, there is a $K \in \mathcal{E}$ such that there are uniformly K-bounded, L-disjoint families ${\mathcal{U}_0}^V, \ldots, {\mathcal{U}_n}^V$ of subsets of $f^{-1}(V)$, whose union covers $f^{-1}(V)$.

Consider the collection

$$\mathcal{W}_{i,j} = \{U^V \colon V \in \mathcal{V}_j, U^V \in {\mathcal{U}_i}^V\}.$$

We claim that this collection (for $j = 0, \ldots, k$ and $i = 0, \ldots, n$) is a K-uniformly bounded, L-disjoint collection of subsets of X that covers X.

Since the $\mathcal{V}_j$ cover Y and the collections ${\mathcal{U}_i}^V$ cover $f^{-1}(V)$, it is clear that the collection $\mathcal{W}_{i,j}$ covers X.

Suppose now that we fix i_0 and j_0 and consider $\mathcal{W}_{i_0,j_0}$. We see that

$$\bigcup_{W \in \mathcal{W}_{i_0,j_0}} (W \times W) = \bigcup_{V \in \mathcal{V}_{j_0}} \bigcup_{U^V \in {\mathcal{U}_{i_0}}^V} \left(U^V \times U^V\right).$$

For each V, the union $\bigcup_{U^V \in {\mathcal{U}_{i_0}}^V} \left(U^V \times U^V\right)$ is a subset of K. Thus, $\bigcup_{W \in \mathcal{W}_{i_0,j_0}} (W \times W)$ is a union of subsets of K and hence a subset of K.

Suppose now that $W \neq W'$ in some $\mathcal{W}_{i,j}$. We can find V and V' in $\mathcal{V}_j$ so that $W \in {\mathcal{U}_i}^V$ and $W' \in {\mathcal{U}_i}^{V'}$. If $V = V'$ then $W \times W' \cap L = \emptyset$ by the assumptions on $\mathcal{V}_j$. In the case that $V \neq V'$, then $W \times W' \cap L \subseteq (f \times f)^{-1}(V \times V') \cap (f \times f)^{-1}(f \times f)(L)$. Since $V \times V' \cap (f \times f)(L) = \emptyset$, we see that $(f \times f)^{-1}(V \times V') \cap L$ is also empty. $\square$

Theorem II.24. *Let* $(X, \mathcal{E})$ *be a coarse space with* $\operatorname{asdim}(X) \leq k$ *and fix* $x_0 \in X$. *Then* $\operatorname{asdim}(*X) \leq 2k + 1$.

Proof. Let T be a graph whose vertex set is in one-to-one correspondence with the elements of $*X$. We connect two vertices t_1 and t_2 of T by an edge if and only if there is an $x \in X$ $(x \neq x_0)$ for which $t_1 x = t_2$ or $T_2 = t_1 x$. It is clear that T is a tree. Give T the bounded coarse structure it inherits as a metric space.

Define $f : *X \to T$ to be the map taking the element $x \in *X$ to the vertex $x \in T$. We use the same symbol for the element and the vertex. In the proof of Theorem II.15, we saw that f is uniformly expansive.

By Lemma II.22, the coarse fibers of f have asymptotic dimension bounded by k uniformly and so, by Lemma II.23 we are done. $\square$

Corollary II.25. *If* B *is a bounded set, then* $\operatorname{asdim} *B \leq 1$.

II.5. Property C and Free Products

For the final section of this chapter, we prove that coarse property C is preserved over coarse properties.

Theorem II.26. *Let (X, E) be a coarse space. Assume that there is a $k \geq 1$ so that for every infinite sequence $E_1 \subseteq E_2 \subseteq \cdots$ of entourages there is a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ of subsets of X such that*

1. *$\bigcup_i \mathcal{U}_i$ covers X;*
2. *$\mathcal{U}_i$ is E_i-disjoint; and*
3. *$\mathcal{U}_i$ has asymptotic dimension bounded by $k-1$ uniformly.*

Then X has coarse property C.

The proof is the same as [BN18, Lemma 6.1]

Definition II.27. A subset $A \subseteq *X$ is said to be **flat** if there is some $\mathbf{x} \in *X$ so that $A \subseteq \mathbf{x} \cdot X^*$.

Let $(X, \mathcal{E})$ be a coarse space with basepoint x_0. Let $E \in \mathcal{E}$ and $A \subseteq *X$. Define the E-**cone** $\operatorname{con}_E(A) = A \cdot (*B(x_0, E))$, where $B(x_0, E)$ is the symmetric ball $\{x \in X \colon (x_0, x) \in E \cup E^{-1}\}$.

Lemma II.28. *Let $\{A_\alpha\}$ be a collection of uniformly bounded flat subsets of $*X$. Then, for each entourage $L \in \mathcal{E}$, the collection $\{\operatorname{con}_L A_\alpha\}$ has asymptotic dimension bounded by 1 uniformly.*

Proof. By assumption there is some $K \in \mathcal{E}$ and an integer n so that $\bigcup_\alpha A_\alpha \times A_\alpha \subseteq \langle K, n \rangle$. Since each A_α is flat, there is (for each α) an $\mathbf{x}_\alpha \in *X$ so that $A_\alpha \subseteq \mathbf{x}_\alpha \cdot X$. Therefore, $A_\alpha \subseteq \mathbf{x}_\alpha \cdot B(x_0, K^n)$ for each α.

Now, $\operatorname{con}_L A_\alpha \subseteq \mathbf{x}_\alpha \cdot B(x_0, K^n) \cdot *B(x_0, L) \subseteq \mathbf{x}_\alpha \cdot *B(x_0, K^n \cup L)$. We apply Corollary II.25 and Lemma II.16 to complete the proof.

$\square$

Definition II.29. Let $(X, \mathcal{E})$ be a coarse space and suppose $E \in \mathcal{E}$. A set $S \subseteq X$ is E**-connected** if for every $x, y \in S$ there is a finite sequence $x = s_0, s_1, \ldots, s_n = y$ of points of S so that $(s_i, s_{i+1}) \in E$ for each i. An E**-connected component** of X is a maximal E-connected subset of X.

Lemma II.30. *Let $(X, \mathcal{E})$ be a coarse space with $x_0 \in X$. Suppose that $E \in *\mathcal{E}$. Take some $L \in \mathcal{E}$ and n so that $E \subseteq \langle L, n\rangle$ and suppose that $A \subseteq *X \cdot (X \setminus B(x_0, L^n))$ has the property that E-connected components of A are uniformly bounded. Then, for each $M \in \mathcal{E}$, the E-connected components of* $\operatorname{con}_M A$ *have asymptotic dimension at most* 1 *uniformly.*

Proof. We prove this first under the assumption that $E = \langle L, n\rangle$. The general case follows from the fact that E-connected components are contained in some $\langle L, n\rangle$-connected component and the fact that asymptotic dimension is monotonic on subsets.

Following the method of [BN18, Lemma 6.11], we can characterize the E-connected components of $\operatorname{con}_M(A)$ as follows: C is an E-connected component of $\operatorname{con}_M A$ if and only if $X^{\leq n} \cap C$ is an E-connected component of $X^{\leq n} \cap \operatorname{con}_M A$.

Let C be an E-connected component of $\operatorname{con}_M A$. Put $C_k = C \cap X^{\leq k}$. Let k_0 be the smallest integer for which $C_{k_0} \neq \emptyset$. We claim that C_{k_0} is flat and uniformly bounded.

Take two words $\mathbf{x}$ and $\mathbf{y}$ in C_{k_0}. Then, $\mathbf{x}$ and $\mathbf{y}$ are in $\operatorname{con}_M A$ and there is an E-chain $(\mathbf{t}_i)$ of elements of C connecting them. By our observation above, we may take the $\mathbf{t}_i$ in C_{k_0}.

Write $\mathbf{t}_1 = \mathbf{a}_t \cdot b_t z_l \cdots z_{k_0}$ with $a_t \cdot b_t \in A$, and $b_t \in X \setminus B(X_0, L^n)$. Similarly, write $\mathbf{x} = \mathbf{a}_x \cdot b_x x_l \cdots x_{k_0}$. Since $(\mathbf{x}, \mathbf{t}_1) \in E$, we have $D_L^*(\mathbf{x}, \mathbf{t}_1) \leq n$. Since $\|b_x\| > n$ and $\|b_t\| > n$, we see that $\mathbf{a}_x = \mathbf{a}_t$, and in particular,

$$n \geq D_L^*(\mathbf{x}, \mathbf{t}_1) = D_L(b_x, b_t) + \sum_{j=l}^{k_0} \left(\|x_j\|_L + \|z_j\|^L\right). \tag{II.1}$$

Next, we suppose that the words $z_l \cdots z_{k_0}$ and $x_l \cdots x_{k_0}$ are non-empty. Then, the minimality of k_0 means that $(\mathbf{a} b_x x_l \cdots x_{k_0 - 1}, \mathbf{x}) \notin E$. Thus, $\|x_{k_0}\|_L > n$, contradicting Equation II.1. We conclude that $C_{k_0} \subseteq A$ and is therefore uniformly bounded. We conclude also that $\mathbf{a}_x = \mathbf{a}_y$ and that C_{k_0} is therefore flat.

We show by induction on k that

$$C_k \subseteq \operatorname{con}_{M \cup L^n \cup D} C_{k_0}.$$

Let $\mathbf{x} \in C_{k+1} \setminus C_k$, with $k \geq k_0$. Then, either $\mathbf{x} \in \operatorname{con}_M C_k$ or $\mathbf{x} \in A$. In the first case, we see that $\mathbf{x} \in \operatorname{con}_{M \cup L^n \cup D} C_k$. In the second case, $\mathbf{x}$ must lie in some E-connected component of A that is also E-close to $\operatorname{con}_M C_k$. Thus, $\mathbf{x} \in \operatorname{con}_{M \cup L^n \cup D} C_k$ as required. Since $\operatorname{con}_{M \cup L^n \cup D} \operatorname{con}_{M \cup L^n \cup D} C_k = \operatorname{con}_{M \cup L^n \cup D} C_k$, we have proved our claim. By Lemma II.28, the E-connected components of $\operatorname{con}_M A$ have asymptotic dimension at most 1 uniformly. $\square$

Theorem II.31. *Let X be a coarse space with fixed basepoint x_0. If X has coarse property C, then $*X$ has coarse property C.*

Proof. Suppose $E_1 \subseteq E_2 \subseteq \cdots$ is a given sequence of entourages in $*\mathcal{E}$. For each i find $L_i \in \mathcal{E}$ and an integer n_i so that $E_i \subseteq \langle L_i, n_i \rangle$. Find a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \mathcal{U}_p$ of $L_i^{n_i}$-disjoint, uniformly bounded subsets of X whose union covers X. Put $n = \max\{n_i\}$.

Put $\mathcal{V}_i(\mathbf{x}) = \{\mathbf{x} \cdot (U \setminus B(x_0, L_{p+1}^n) \colon U \in \mathcal{U}_i, \mathbf{x} \in *X\}$ for each $i \in \{1, 2, \ldots, p\}$.

Put $\mathcal{V}_{p+1} = \{\{x_0\}\}$. We claim that

1. $\bigcup_{i=1}^{p+1} \mathrm{con}_{L^n_{p+1}} \cup \mathcal{V}_i = *X$;

2. $\{\mathcal{V}_i(\mathbf{x})\}_{i,\mathbf{x}}$ is E_i-disjoint, uniformly bounded, and its elements are flat.

For (1), we consider an element $\mathbf{x} \in *X$. Write $\mathbf{x} = x_1 x_2 \cdots x_k$, where each $x_i \in X$. If $(x_0, x_i) \in L^n_{p+1}$ for each i, then $\mathbf{x} \in \mathrm{con}_{L^n_{p+1}}\{x_0\}$. Otherwise, take m to be the largest integer for which $(x_0, x_m) \notin L^n_{p+1}$. Find some U in some $\mathcal{U}_l$ so that $x_m \in U$. Then, $x \in x_1 x_2 \cdots x_{m-1} \cdot (U \setminus B(x_0, L^n_{p+1})) \cdot *B(x_0, L^n_{p+1})$. Thus, $\mathbf{x} \in \mathrm{con}_{L^n_{p+1}} \cup \mathcal{V}_l$.

For (2), we observe that each $\mathcal{V}_i$ is uniformly bounded and flat, so it remains only to show that these families are E_i-disjoint. Suppose that V_1 and V_2 are distinct elements of some $\mathcal{V}_i$. We can find $\mathbf{x}_1, \mathbf{x}_2 \in *X$ and subsets U_1 and U_2 in $\mathcal{U}_i$ for which $V_1 = \mathbf{x}_1 \cdot \tilde{U_1}$ and $V_2 = \mathbf{x}_2 \cdot \tilde{U_2}$, where $\tilde{U_i}$ denotes the set U_i with the ball $B(x_0, L^n_{p+1})$ removed. If $\mathbf{x}_1 = \mathbf{x}_2$, then we're done since $\mathcal{U}_i$ is $L_i^{n_i}$-disjoint.

Otherwise, take $\mathbf{v}_1 = \mathbf{x}_1 u_1$ and $\mathbf{v}_2 = \mathbf{x}_2 u_2$. We rewrite $\mathbf{v}_1 = (\mathbf{v}_1 \wedge \mathbf{v}_2) b \mathbf{c} u_1$ and $\mathbf{v}_2 = (\mathbf{v}_1 \wedge \mathbf{v}_2) b' \mathbf{c}' u_2$. We compute

$$\begin{aligned} D^*_{L_i}(\mathbf{v}_1, \mathbf{v}_2) &= D_{L_i^{n_i}}(b, b') + \|\mathbf{c}\|_{L_i} + \|\mathbf{c}'\|^{L_i} + D_{L_i}(x_0, u_1) + D_{L_i}(u_2, x_0) \\ &\geq D_{L_i^{n_i}}(b, b') + \|\mathbf{c}\|_{L_i} + \|\mathbf{c}'\|^{L_i} + D_{L_i}(u_1, u_2) \\ &\geq n_i. \end{aligned}$$

Thus we see $\mathcal{V}_i$ is $\langle L_i, n_i \rangle$-disjoint hence E_i-disjoint..

□

CHAPTER III

PERSISTENCE CURVES

III.1. Introduction

Topological data analysis (TDA) is a relatively new field of mathematics that seeks to examine the shape and structure of data. Persistent homology (PH) is an important tool in TDA developed in 2002 [ELZ00] based on the work of size functions in the 1990's [Fro92, FFLZ98]. Since its inception, TDA has permeated through many disciplines. Indeed applications of TDA can be found in neuroscience [BMM+16], medical biology [LCG+15], sensor networks [DSG+07], social networks [CH13], physics [DGP+16], computation [LGZ16], nanotechnology [NHH+15], and more.

Persistent homology transforms a data set into a sequence of topological spaces, called a filtration, where it tracks when features, such as holes and components, appear (are born) and disappear (die). Collecting this birth-death information leads to a visual summary called a persistence diagram. Intuitively, the amount of time a hole or component exists, called the lifespan, indicates the relative importance of the associated feature. For many applications, this intuition holds though it is not always necessarily the case [BMM+16]. By using the bottleneck distance, one can consider the metric space of persistence diagrams. We cannot easily use machine learning methods or other common data analysis methods on this metric space as it has a multi-set structure and was shown to have very little other structure [MMH11]. However, there exist methods to transform persistence diagrams into something more pallatable for these analytics methods.

III.2. Recent Work

Summarizations of persistence diagrams can be placed into two main categories: kernel estimation and vectorization. In the former, one constructs a kernel function, or a rule for measuring and quantifying the likeness of two persistence diagrams. This method has been seen through a bag-of-words [LOC14] and kernel SVM for persistence [RHBK15]. To vectorize a persistence diagram is to map it into a certain Hilbert space. This summarization type has been well-studied in the forms of Persistence Landscapes [Bub15], Persistence Images [AEK$^+$17], Persistent Entropy [AGS18], Euler Characteristic Curve[RW14], etc. Persistence Landscapes provides a stable (which we will define in Section III.4.3) functional representation of a diagram, which maps a diagram to an element in L^2. PL has many applications. Persistence Images [AEK$^+$17] provides a stable way to transform persistence diagrams through the placement of small surfaces over each diagram point. Persistent Entropy defines an entropy derived from the information theory, and provides a stable summary of persistence diagrams. Euler Characteristics Curve, and Betti number curve have been studied and used before the theory of persistent homology was developed. These two summarization methods are not mutually exclusive and have been combined via Persistence Codebooks [ZJZ18]. We present a vectorization method, and apply it to texture analysis.

Texture analysis is a fundamental task in many scientific areas, such as image processing, material science [Bun13], geology [RE14], brain disease [HGCC$^+$17], thyroid nodules [KKM$^+$15], and more. There have been studies on texture classifications by TDA tools [LOC14, GMBB18, CNO18, RHBK15]. Our approach is different from those, and is simpler to implement. Most importantly, in many cases our approach outperforms those results. Adams, et. al [AEK$^+$17] neatly outline the qualities of a good vectorization method as follows:

Problem. How can we represent a persistence diagram so that

1. The output of the representation is a vector in $\mathbb{R}^n$
2. The representation is stable with respect to the input noise
3. The representation is efficient to compute
4. The representation maintains an interpretable connection to the original persistence diagram
5. The representation allows one to adjust the relative importance of points in different regions of the persistence diagram

III.3. Contribution

Presented in this work is the vectorization method called Persistence Curves. This summary is based on the Fundamental Lemma of Persistent Homology. In Sections III.4 we provide necessary background from the field of TDA, namely cubical homology and persistent homology. In SectionIII.5 we present such a vectorization called persistence curves. The general definition of persistence curves allows them to readily output real vectors of any size hence making them very compatible with machine learning techniques. By making natural choices, we will see persistence curves carry interpretable information about the diagram and hence the underlying space from which the diagram arises. We also show that the representation allows one to adjust relative importance based on regions of a diagram. Finally, we prove a general stability theorem and a useful Corollary. In Section III.6, we provide experimental evidence of the computational efficiency of these curves and compare the performance of various persistence curves with other TDA methods on four popular texture databases: Outex [OMP+02], UIUCTex[PNS12], KTH Texture under varying illumination, pose, and scale (KTH-TIPS) 2[MTMF+], and the Flickr Material Database (FMD)[SRA14]. With Section III.7, we end by presenting many possible avenues for the advancement of persistence curves.

III.4. Background

We introduce and discuss necessary notions and notations of algebraic topology in this section. We begin with cubical complexes and homology, and persistent homology. Much of this section follows the discussion in [DW16]. For a more detailed introduction, see, for example, [KMM04] for cubical homology and [EH10] for persistent homology.

III.4.1. Binary, Gray-scale, and Color Images

Our main application is to texture analysis via gray-scale and color images. A cubical complex is one of the most intuitive ways to represent and study images. For that reason, we review the theory of cubical complex and homology in this section. It should be noted that cubical homology is just one of many possible homology theories for persistence. The method of persistent homology does not depend on the type of homology theory chosen. Hence, the method of persistence curves will not depend on this choice either.

Formally, an $n \times m$ **grayscale image** is a function $I : \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots 255\}$. We associate grayscale images as a collection of tiny squares called pixels each with a shade of gray called the pixel values. In this case, the pixel in the i-th row and the j-th column has a pixel value of $I(i, j)$. One may view a binary image as an image whose codomain is the set $\{0, 1\}$. In this case we associate 0 with the color black and 1 with the color white.

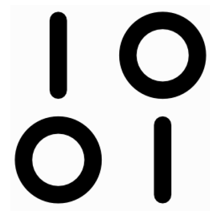
For a grayscale image $I(i, j)$, we construct the **binary image** as the sublevel set of I

$$I_t(i, j) = \{(i, j) \mid I(i, j) \leq t\}, \text{ for } t = 0, 1, \ldots, 255.$$

III.4.2. Cubical Homology

Homology provides a discrete object as a descriptor of a topological space. As a topological invariant, the homology of a topological space is stable under continuous deformations. This means that the homology of a space gives us useful information about its structure. Though homology is not as strong as invariants like homotopy or homeomorphisms, the complexity (or lack thereof) of its computability makes it a feasible tool for applications. Recall that binary images are the collection of squares from which we obtain a topological summary by counting clusters of white pixels (connected components) and clusters of black pixels (holes). These numbers are called *Betti* numbers. Although the formal definition of Betti numbers will be given at the end of this section, their intuition can be described in a concrete example as shown in Figure III.1. In particular, in this work, we are interested in shapes built up by white pixels inside binary images. In Figure III.1(a), we have a binary image that has 4 white regions, and hence, the 0-th level Betti number is 4, i.e. $\beta_0 = 4$. Moreover, there are two black regions that were enclosed by the white regions, and hence, the first level Betti number is 2, i.e. $\beta_1 = 2$. In some cases, one may want to consider the shapes built up by black pixels. In this case, one may take the complement of the binary image (interchanging the colors) as shown in Figure III.1(b). Notice however doing so does not simply flip β_0 and β_1. This means the complement image, gives additional information and we will see that information is worth considering.

(a) The Betti numbers of a binary image. By convention a binary image represents the cubical complex X of white pixels in the image, and its Betti number is $\beta_0(X) = 4$ and $\beta_1(X) = 2$. Note that if the image is surrounded a boundary of white pixels, then $\beta_0(X) = 5$ and $\beta_1(X) = 3$.



(b) The image represents the cubical complex X that arises from the complement of the image in (a). Its Betti numbers are $\beta_0(X) = 3$ and $\beta_1(X) = 4$ whether or not there is an assumed boundary of white pixels.

Figure III.1. Betti Numbers and the Boundary Effect.

To understand Betti numbers formally, we consider intervals of the form $[\ell, \ell+1]$ or $[\ell, \ell] := [\ell] = \{\ell\}$ where $\ell \in \mathbb{Z}$, these are called **elementary intervals**. Intervals of the form $[\ell]$ are called **degenrate**. We define an **elementary cube** to be a finite product of such intervals. In other words, Q is an elementary cube if $Q = J_1 \times J_2 \times \ldots \times J_n$ where J_j is an elementary interval for $j = 1, \ldots k$. We let $\mathcal{K}$ represent the collection of all elementary cubes.

We say the set X is **cubical** if it can be written as a finite union of elementary cubes. For example, one may consider pixels as elementary cubes. More precisely, let (i, j) be a pixel of a 2D binary image. We view (i, j) as an elementary cube $[i, i+1] \times [j, j+1]$. Binary images are unions of cubes of $[i, i+1] \times [j, j+1]$ types, and hence, binary images are cubical. By way of notation we define the set $\mathcal{K}(X) = \{Q \in \mathcal{K} \mid Q \subseteq X\}$.

Now that we have a basic topological framework, we seek to provide a complementary algebraic framework. For this, we fix a ring R and cubical set X. We define the **k-th chain module** over X, denoted $C_k(X)$ to be the formal span of its elementary cubes of dimension k. In other words, by defining $\mathcal{K}_k(X) = \{Q \in \mathcal{K}(X) \mid \dim Q = k\}$, where $\dim Q$ is the number of non-degenerate intervals in Q, we see

$$C_k(X;R) = \left\{ \sum_{Q \in \mathcal{K}_k(X)} \alpha_Q Q \colon \alpha_Q \in R \right\}.$$

Note that since $\mathcal{K}_k(X)$ is finite, this defines an R-module for each $k \in \mathbb{N}$. Each element of a chain module is called a **chain**. The **support** of a chain is simply the elementary cubes with nonzero coefficients. Naturally then, we would like some way to connect these chain modules over all dimensions. To do this, we need the **algebraic boundary map**, ∂. We define this map in pieces. For any interval $[\ell, \ell+1]$, $\partial([\ell, \ell+1]) := [\ell+1] - [\ell]$ and $\partial([\ell]) = 0$ for every degenerate interval. Now that we understand the map for intervals, we define ∂ for elementary cubes:

$$\partial(J_1 \times J_2 \times \ldots \times J_m) = \sum_{j=1}^{m} J_1 \times \ldots \times \partial(J_j) \times \ldots J_m,$$

where if $J_j = [\ell, \ell+1]$ we have

$$J_1 \times \ldots \times \partial(J_j) \times \ldots J_m = J_1 \times \ldots \times [\ell+1] \times \ldots J_m - J_1 \times \ldots \times [\ell] \times \ldots J_m,$$

and in the case J_j is degenerate, we obtain 0. he fundamental proposition of boundary maps is the following.

Proposition III.1. *[KMM04] For any elementary cube Q, one has $\partial\partial Q = 0$.*

In fact, this operator extends naturally to a linear operator on the chain modules. In particular, given a chain $c = \sum_{i=1}^{m} \alpha_i Q_i$ we see $\partial(c) = \sum_{i=1}^{m} \alpha_i \partial(Q_i)$. Thus we see $\partial\partial c = 0$. Notice also a single application of ∂ reduces the embedding dimension by at least 1. This means now $\partial \mid_{C_k(X;R)} := \partial_k : C_k(X;R) \to C_{k-1}(X;R)$ is a map from the k-chain module to the $(k-1)$-chain module. Thus given a cubical set

X we can construct the **chain complex** $\mathcal{C}(X;R) = \{(C_k(X;R), \partial_k) \mid k \in \mathbb{N}\} \cup \{0\}$ which is the collection of all k-chain modules along with their boundary maps along with the zero space. To avoid notation overload, we will write C_k for $C_k(X;R)$ when the context is clear. We can realize a chain complex as the following sequence

$$\ldots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Now we call the kernel, $\ker \partial_k$, of the k-th boundary map to be the **k-cycles** or just **cycles** and the image $\operatorname{im} \partial_k$ to be the $k+1$-boundary or just **boundary**. Notice that because $\partial\partial \equiv 0$ we have $\operatorname{im} \partial_k$ is a normal additive subgroup of $\ker \partial_{k+1}$. Hence we define the $k-th$ homology group of $\mathcal{K}$ to be

$$H_k(X;R) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

This gives rise to the mantra: "homology equals cycles mod the boundaries." We call the rank (or order) of the k-th homology group the **k-th Betti number** and write $\beta_k(X;R)$. Intuitively, the k-th Betti number tells us how many k-dimensional holes our underlying space has. The 0-th level Betti number counts the number of connected components, the first level Betti number counts the number of "circles" we have and the second level Betti number counts the number of "air pockets" or "spheres" the space has and so on.

III.4.3. Persistent Homology

At this point, we see the formal definition of Betti number and how to count them for a given set. We cannot directly compute or define the homology of a grayscale or color image, since grayscale or color images are functions rather than sets. One quick idea would be to use thresholding as seen in Equation (III.4.1). However, this would force us to create a universal rule for selecting a threshold. Instead, we turen to persistent homology, which allows us to consider every threshold at once by creating a series of spaces related by inclusion.

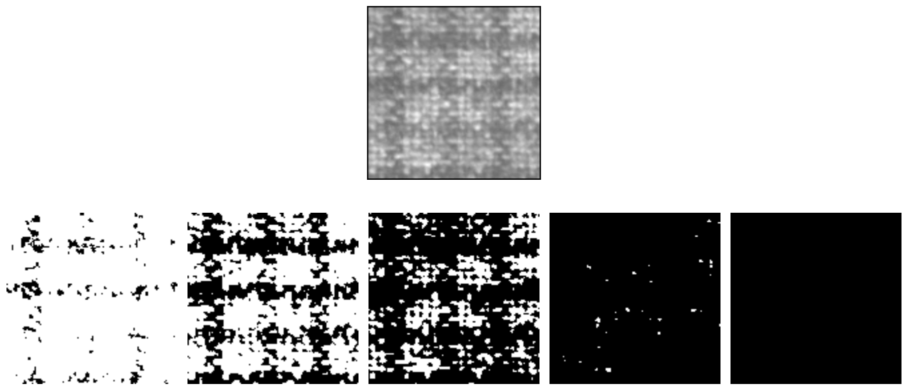


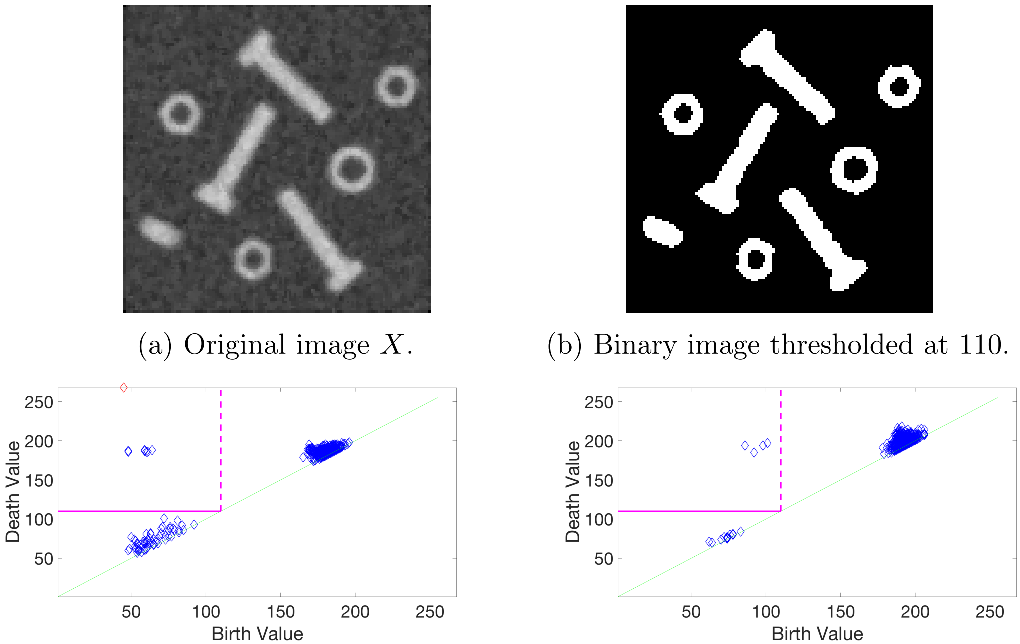
Figure III.2. A Filtration of an Outex Image

Suppose X represents a cubical set and let $\mathcal{K}$ represent the corresponding cubical complex. We define a function $f : \mathbb{R} \to \mathcal{K}$ and require that whenever $a \leq b$, we have $f(a) \subseteq f(b)$. Such a function is called a **filtering function**. Let $(a_1, a_2, \ldots, a_n)$ be a finite increasing sequence of real numbers where $f(a_n) = \mathcal{K}$. Then a **filtration** of $\mathcal{K}$ is the corresponding sequence $f(a_1) \subseteq f(a_2), \subseteq \ldots \subseteq f(a_n)$. We create a filtering function for an image by the function $f(t) \mapsto \mathcal{K}(I_t)$.

Suppose that we have a filtration of a cubical complex, $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \ldots \subseteq \mathcal{K}_n$. The inclusion induces homomorphism between the homology groups so that for each k we have $H_k(\mathcal{K}_1) \xrightarrow{i_1^*} H_k(\mathcal{K}_2) \xrightarrow{i_2^*} \ldots \xrightarrow{i_{n-1}^*} H_k(\mathcal{K}_n)$, where each i_j^* is a homomorphism. We say a homology class α is **born** at j if we have $\alpha \in H_k(\mathcal{K}_j)$ and $\alpha \notin i_{j-1}^*((\mathcal{K}_{j-1})$. We say α dies at $H_k(\mathcal{K}_j)$ if $\alpha \in H_k(\mathcal{K}_{j-1})$ and one of the following hold: $i_{j-1}^*(\alpha)$ is trivial; or if α is born at j and β is born at $\ell < j$ and $f_{i^*-1}(\alpha) = f_{i^*-1}(\beta)$.

In the last condition we employ the **elder rule**, which allows us to uniquely define the death of a class. This rule says in the choice between two classes, we choose to keep the "oldest" class. We can guarantee that every homology class $\alpha \in H_k(\mathcal{K}_i)$ for some i has a birth time. We cannot guarantee that each class has a death time. For such classes, we assign the "death time" as ∞. This procedure allows us to define a unique set of points for each homology class (b, d) where b is the birth time of the class and d is its death time. We will define the persistence diagram, but first let us recall the concept of a multi-set. A **multi-set** is a set S along with a **multiplicity function** $M : S \to \mathbb{N} \cup \{\infty\}$, we denote a multi-set by the tuple (S, M). Suppose we have a filtering function f of a cubical set X. A **persistence diagram** is a multi-set $\mathcal{P}_k(X, f) = (P_k, M_k)$ where P_k consists of all unique birth-death pairs along with the diagonal $\{(x, x) \mid x \in \mathbb{R}\}$ and M_k assigns the multiplicity of the birth-death pairs and $M_k(x, x) = \infty$ for each diagonal element. An example of a diagram can be found in Figure III.3(c)-(d). We will often denote $\mathcal{P}_k(X, f)$ by $\mathcal{P}_k$ when the context is obvious. Figure III.3 shows an example of persistence diagrams of a grayscale image. Although Figure III.3(a) is a grayscale image, visual inspection suggests that there are 8 "white" pieces and 4 "black" holes. Its persistence diagrams as shown in Figure III.3(c)-(d) confirms the intuition. There are 8 (4) points that are away from the diagonal line in $\mathcal{P}_0$ ($\mathcal{P}_1$), and that suggests they persist for a long period of thresholds. Hence, they are likely the robust features.

We would like to know that this summary of our data is stable in some sense. For us, this means that if we perturb the original points by a small amount, we want the diagrams to different by a small amount. This so-called stability theorem is the cornerstone of persistent homology. One must then ask the question, "how can we measure the difference of persistence diagrams?"



(a) Original image X.

(b) Binary image thresholded at 110.

(c) $\mathcal{P}_0(X)$.

(d) $\mathcal{P}_1(X)$.

Figure III.3. Example of Fundamental Lemma of Persistent Homology

To answer this question we turn our attention to the **Wasserstein p-metric** W_p for $1 \leq p \leq \infty$, which we define as follows. Given two diagrams $D_1 = \mathcal{P}_k(X, f)$ and $D_2 = \mathcal{P}_k(Y, g)$

$$W_p(D_1, D_2) = \inf_{\substack{\text{bijections} \\ \eta : D_1 \to D_2}} \left(\sum_{x \in D_1} \|x - \eta(x)\|^p \right)^{1/p},$$

for $1 \leq p < \infty$ and

$$W_\infty(D_1, D_2) = \inf_{\substack{\text{bijections} \\ \eta : D_1 \to D_2}} \sup_{x \in D_1} \|x - \eta(x)\|.$$

The Wasserstein ∞-metric is also known as **bottleneck distance**. The Wasserstein distance searches over all possible pairings to find an optimal one. It should be noted that an optimal pairing is not necessarily unique. Given the necessity of the bijections

η, it now becomes apparent, why we need the diagonal to have infinite multiplicity. Otherwise, a pairing would not be possible. This brings us to the popular stability theorem that states that the bottleneck distance is 1-Lipschitz with respect to the ∞-norm on the filtering functions. That is to say if X is a cubical set then

$$W_\infty(\mathcal{P}_k(X, f), \mathcal{P}_k(X, g)) \leq \|f - g\|_\infty.$$

The proof of which first appeared in [CSEH07]. We end this section with a note that cubical homology and sublevel sets are just one example of many processes by which persistence diagrams arise. In the following section, we will present persistence curves, which rely on persistence diagrams and not the process by which they were created.

III.5. Persistence Curves and Stability

This section contains the main idea to construct the persistence curves. The essential idea is the Fundamental Lemma of Persistent Homology [EH10] which is derived from the elder rule. This lemma states that given a filtering function f on a space X, one has the following calculation for the corresponding k-dimensional diagram D_k,

$$\beta_k(f(t)) := \beta(t) := \#\{(b, d) \in D_k \mid b \leq t, d > t\}, \tag{III.1}$$

where # represents the counting measure. Figure III.3(c)-(d) shows an example of (III.1). In particular, the rectangular boxes in (c)-(d) enclosed by the dotted line represents the set $\{(b, d) \in D_k \mid b \leq 110, d > 110\}$ for $k = 0,\ 1$, respectively. It is straightforward to observe that $\#\{(b, d) \in D_0 \mid b \leq 110, d > 110\} = 8$ and $\#\{(b, d) \in D_1 \mid b \leq 110, d > 110\} = 4$, which are the Betti numbers of Figure III.3(b) by visual inspection. However, (III.1) does not fully utilize the information of the set.

Thus we seek to generalize (III.1) in the following way. Let $\mathcal{D}$ represent the set of all persistence diagrams. let $\mathcal{F}$ represent the set of all functions $\psi : \mathcal{D} \times \mathbb{R}^3 \to \mathbb{R}$ so that $\psi(D; x, x, t) = 0$ for all $x \in \mathbb{R}$. Let $\mathcal{T}$ represent the set of **statistics** or operators on multi-sets, and finally let $\mathcal{R}$ represent the set of functions on $\mathbb{R}$.

Definition III.2. We define a map $P : \mathcal{D} \times \mathcal{F} \times \mathcal{T} \to \mathcal{R}$ where

$$P(D, \psi, T)(t) = T(\{\psi(D; b, d, t) \mid \ b \leq t, \ d > t\}).$$

The function $P(D, \psi, T)$ is called a **persistence curve** on D with respect to ψ and T. To better understand persistence curves, we begin by discussing examples.

Example III.3. Let $\mathbf{1}(x, y, t) = 1$ if $x \neq y$ and 0 otherwise. Let T be the summation operator, Σ. By Definition III.2, we have

$$\begin{aligned} P(D, \mathbf{1}, \Sigma) &= \sum(\{\mathbf{1}(D; b, d, t) \mid \ b \leq t, \ d > t\}) \\ &= \#\{(b, d) \in D \mid b \leq t, d > t\} = \beta(t). \end{aligned}$$

Suppose D_k represents the k-dimensional diagram associated to a given filtration $\mathcal{X}$. Using the Fundamental Lemma of Persistent Homology, we may define the **Euler Characteristic** of the space X_t corresponding to a threshold t is the alternating sum of the space's Betti numbers. That is,

$$EC(X_t) := EC(t) = \sum_{i=0}^{\infty}(-1)^i \beta_i(t) = \sum_{i=0}^{\infty}(-1)^i P(D_i, \mathbf{1}, \Sigma)(t).$$

The term **Euler Characteristic Curve** with respect to the filtration $\mathcal{X}$ refers to the function

$$ECC(X) \equiv \sum_{i=0}^{\infty}(-1)^i P(D_i, \mathbf{1}, \Sigma).$$

Example III.4. Let $\psi = d-b$ and T be the Σ. Then the $P(D,\psi,\Sigma) = \sum(\{d-b \mid b \leq t,\ d > t\})$. Figure III.4 illustrates $P(D,\psi,\Sigma)$. Figure 4 shows the a life curve and at 5 selected threshold values, we see the set $\{d-b \mid b \leq t,\ d > t\}$. The value of the curve at that threshold if calculated by applying ψ to each highlighted point and summing over all such function values.

Here are some examples that are independently studied in different contents, but they can be realized as persistence curves.

Example III.5. *[CD18] developed an optimal thresholding method in imaging processing based on persistence diagrams. The main idea in [CD18] is to define an objective function, and the optimal threshold will be chosen at the maximum of the objective function. One major component of the objective function in [CD18] is* $O(t) = \frac{1}{\#D(t)} \sum_{(b,d)\in D(t)} (d-t)(t-b)$*, where* $D(t) = \{d-b \mid b \leq t,\ d > t\}$*.* $O(t)$ *can be viewed as persistence curves if one lets* $\psi = (d-t)(t-b)$ *and* T *be the average operator, then* $P(D,\psi,T) = O(t)$*.*

Another example is by [AGS18], where the concept of entropy was introduced to TDA.

Example III.6. In [AGS18], a summary function based on persistence entropy was defined $S(D)(t) = -\sum w(t)\frac{d-b}{L}\log(\frac{d-b}{L})$, where $L = \sum(d-b)$ and $w(t) = 1$ if $b \leq t \leq d$ and $w(t) = 0$ otherwise. Let $\psi = -\frac{d-b}{\sum(d-b)}\log\frac{d-b}{\sum(d-b)}$, and $T = \Sigma$. Then we find that $E(D) := P(D,\psi,T)$ is similar to $S(D)$. In fact due to the exclusion of the death value in the interval we have $0 \leq E(D) \leq S(D)$. Hence $E(D)$ enjoys the same stability as proven in [AGS18].

In this last example, we also recognize persistence landscapes, a well-known diagram summary, as a special case of persistence curves.

Example III.7. Let $\max_k(S)$ represent the k-th largest number of a set S. [Bub15] defines the following functions for a diagram D.

$$f_{(b,d)}(t) = \begin{cases} 0 & \text{if } t \notin (b,d) \\ t-b & \text{if } t \in (b, \frac{b+d}{2}] \\ d-t & \text{if } t \in (\frac{b+d}{2}, d) \end{cases}$$

Then the k-th Persistence Landscape [Bub15] is defined by $\lambda_k(t) = \max_k\{f_{(b,d)}(t) \mid (b,d) \in D\}$. We see then with $\psi(b,d,t) = \min\{t-b, d-t\}$ and $T = \max_k$, $P(D,\psi,T) \equiv \lambda_k$. Indeed one can quickly see through direct calculation that $\psi \cdot w(b,d,t) \equiv f_{(b,d)}(t)$. We see that under the Persistence Curve framework, two seemingly disparate object, namely the Betti curve and persistence landscapes, are connected.

Although we have the ability to choose any function of the diagrams, we tend to choose the functions that carry sensible information about the diagram and hence the underlying space. Many of these curves are built from well-studied persistence diagram statistics. Table III.1 lists a few persistence curves that rely on well-studied diagram statistics. For example, the **Midlife** quantity has been used in recent work, such as persistence landscapes [Bub15] and persistence image [AEK$^+$17] to serve as a linear transformation. The **Multiplicative Life** quantity has been studied in the field of random complexes [BKS$^+$17]. The life-entropy persistence curve actually appears as the entropy summary function in a recent paper by Atienza, Gonzales-Diaz, and Soriano-Trigueros [AGS18]. We also introduce two new entropy-like functions using the multiplicative life and midlife statistics.

Table III.1. Examples of Persistence Curves.

Name	**Notation**	$\psi(b,d,t)$	T
Betti	$\boldsymbol{\beta(D)}$	1	sum
Midlife	$\mathbf{ml}(D)$	$(b+d)/2$	sum
Life	$\boldsymbol{\ell}(D)$	$d-b$	sum
Multiplicative Life	$\mathbf{mul}(D)$	d/b	sum
Life Entropy [AGS18]	$\mathbf{le}(D)$	$-\frac{d-b}{\sum(d-b)}\log\frac{d-b}{\sum(d-b)}$	sum
Midlife Entropy	$\mathbf{mle}(D)$	$-\frac{d+b}{\sum(d+b)}\log\frac{d+b}{\sum(d+b)}$	sum
Mult. Life Entropy	$\mathbf{mule}(D)$	$-\frac{d/b}{\sum(d/b)}\log\frac{d/b}{\sum(d/b)}$	sum
k-th Landscape [Bub15]	$\boldsymbol{\lambda_k}(D)$	$\min\{t-b,d-t\}$	$\max_k$

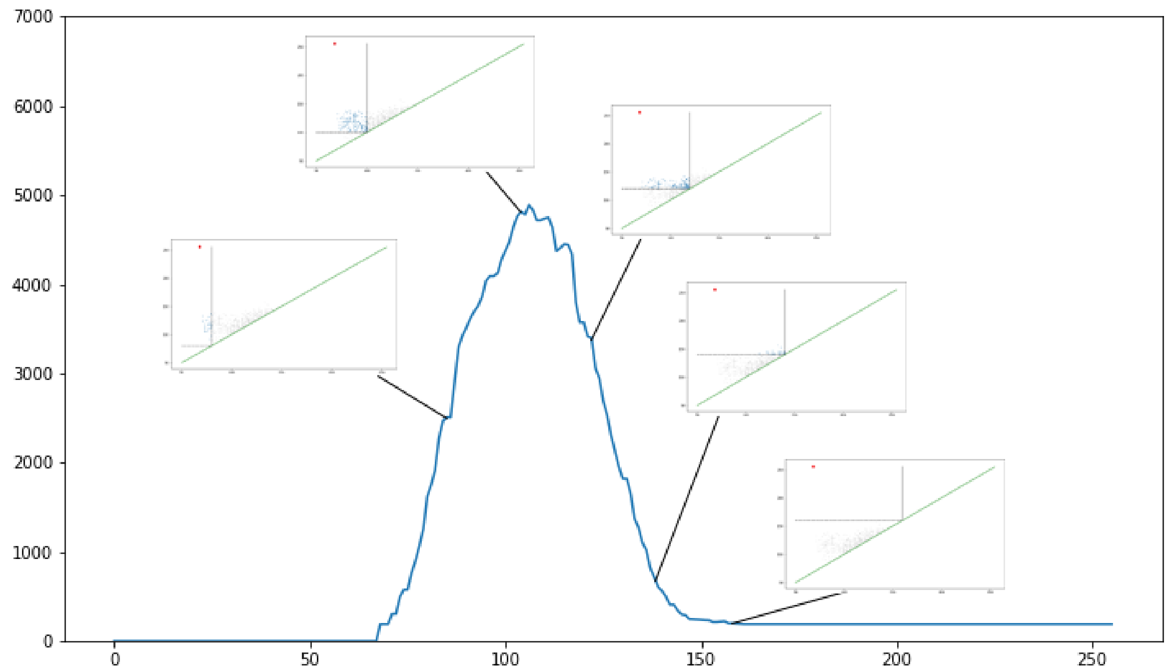


Figure III.4. A Persistence Curve and Diagrams at the Correspondoing Threshold Values

Table III.2. Notation for Diagrams C and D Matched Optimally for Bottleneck Distance

Notation	**Description**
n	The maximum number of off diagonal points between each diagram
(b_i^D, d_i^D)	a point in persistence diagram D
ψ_i^D	$\psi(b_i^D, d_i^D)$
Ψ^D	$\sum \psi_i^D$
$w_i^D(x)$	$w(b_1, d_i, x) = 1$ if $b_i^D \leq x \leq d_i^D$; 0 otherwise.
L^D	$\sum d_i^D - b_i^D$

Before our next theorem, we will set up some notation and conventions. Let $C, D \in \mathcal{D}$. Let n represent the maximum between the number of off diagonal points in each diagram and note that n is finite. We assume the optimal matching under the bottleneck distance of these diagrams is known and we index the points of each diagram $\{(b_i, d_i)_D\}_{i=1}^n$ and $\{(b_i, d_i)_C\}_{i=1}^n$ so that points with matching indices are paired under the optimal matching. We define $L^D = \sum_{i=1}^n (d_i - b_i)$ and analogously for C. If $\psi \in \mathcal{F}$, we define $\psi_i^D = \psi(b_i, d_i)$ for $i = 1, \ldots n$, $\Psi^D = \sum_{i=1}^n \psi(b_i, d_i)_D$, and $w_i \equiv w(b_i, d_i, \cdot)$ again, analogously for C. Notice that if T is the sum statistic, then $P(D, \psi, T)(t)$ can be written as $\sum_{(b,d) \in D} \psi(b, d, t) w(b, d, t) = \sum_{i=1}^n \psi_i^D w_i^D$. For easy reference, this notation is summed up in Table III.2. Finally, in regards to infinite death values one has a couple options. If there is a global maximum finite death value for the space, we set all infinite death values to this maximum. For example, in the case of images, 255 is the max pixel value, hence the largest possible finite death value.

If no such value exists, we can set the infinite death values to be equal to the largest finite death value in the current filtration.

Theorem III.8. *Let $C, D \in \mathcal{D}$ and index them through the optimal bottleneck distance matching. Let T be the sum statistic. We adopt the notations in Table III.2. Let $\psi \in \mathcal{F}$ and*

$$\varepsilon := \min\{L^C \max_i |\psi_i^C - \psi_i^D| + 2\Psi^D W_\infty(C, D), L^D \max_i |\psi_i^C - \psi_i^D| + 2\Psi^C W_\infty(C, D)\}.$$

Then we have

$$\|P(C, \psi, T) - P(D, \psi, T)\|_1 \leq \varepsilon.$$

Proof. We are interested in the difference.

$$\|P(C, \psi, T) - P(D, \psi, T)\|_1 = \left\| \sum_{i=1}^n \psi_i^D w_i^D - \sum_{i=1}^n \psi_i^C w_i^C \right\|_1$$

This becomes

$$\begin{aligned}
\|P(D, \psi, T) - P(C, \psi, T)\|_1 &= \left\| \sum_{i=1}^n \psi_i^D w_i^D - \sum_{i=1}^n \psi_i^C w_i^C \right\|_1 \\
&= \left\| \sum_{i=1}^n \psi_i^D w_i^D - \psi_i^C w_i^C \right\|_1 \\
&\leq \sum_{i=1}^n \left\| \psi_i^D w_i^D - \psi_i^C w_i^C \right\|_1 \\
&\leq \sum_{i=1}^n \|w_i^C\|_1 |\psi_i^C - \psi_i^D| + |\psi_i^D| \|w_i^C - w_i^D\|_1.
\end{aligned}$$

We'll focus on each summand individually. Notice that w_i is simply the indicator function on the interval $[b_i, d_i)$, hence its norm is just the lifespan of the corresponding diagram point, $d_i - b_i$. Thus we see for the first,

$$\sum_{i=1}^{n} \|w_i^C\|_1 |\psi_i^C - \psi_i^D| \leq \sum_{i=1}^{n} \|w_i^C\|_1 \max_i |\psi_i^C - \psi_i^D| \leq L^C \max_i |\psi_i^C - \psi_i^D|.$$

The dichotomous nature of w_i allows us to write $w_i^D = w_i^D w_i^C + w_i^D(1 - w_i^C)$ and vice versa for w_i^C. For the second summand we note that $\|w_i^C - w_i^D\|_1 \leq \|w_i^D(1 - w_i^C)\|_1 + \|w_i^C(1 - w_i^D)\|_1$ and $w_i^D(1 - w_i^C) = 1 \Rightarrow w_i^C(1 - w_i^D) = 0$. In consideration of the pairing $(b_i^C, d_i^C), (b_i^D, d_i^D)$ there are three cases to consider up to permutation. Those cases are

1) $b_i^C \leq d_i^C \leq b_i^D \leq d_i^D$;

2) $b_i^C \leq b_i^D \leq d_i^c \leq d_i^D$;

3) $b_i^C \leq b_i^D \leq d_i^D \leq d_i^C$.

In each case, it is straightforward to observe that

$$\|w_i^D(1 - w_i^C)\|_1 + \|w_i^C(1 - w_i^D)\|_1 \leq 2 \cdot \max_i \{|b_i^C - b_i^D|, |d_i^C - d_i^D|\} \leq 2 \cdot W_\infty(C, D).$$

Thus, we have

$$\sum_{i=1}^{n} |\psi_i^D| \|w_i^C - w_i^D\|_1 \leq \sum_{i=1}^{n} |\psi_i^D| (2 \cdot \max_i \{|b_i^C - b_i^D|, |d_i^C - d_i^D|\}) \leq 2\Psi^D W_\infty(C, D).$$

Hence, we obtain

$$\|P(D, \psi, T) - P(C, \psi, T)\|_1 \leq L^C \max_i |\psi_i^C - \psi_i^D| + 2\Psi^D W_\infty(C, D).$$

However, we must note that there was a choice of splitting in the beginning. Therefore, with

$$\varepsilon = \min\{L^C \max_i |\psi_i^C - \psi_i^D| + 2\Psi^D d_\infty(C,D), L^D \max_i |\psi_i^C - \psi_i^D| + 2\Psi^C W_\infty(C,D)\},$$

we conclude

$$\|P(D,\psi,T) - P(C,\psi,T)\|_1 \leq \varepsilon.$$

□

Note that there is no assumption on ψ. Theorem III.8 offers a general bound on persistence curves associated with different persistence diagrams. It does not reveal the stability (yet). As we will discuss in Section III.6, unstable persistence curves may also serve reveal important and useful topological information, such as ECC. On the other hand, if the stability is of the interest, with an additional assumption on ψ, the stable persistence curves can be obtained as stated below.

Corollary III.9. *Under the same assumptions as in Theorem III.8, if we further assume that ψ is a stable measurement, i.e. $\|\psi^C - \psi^D\| \leq KW_\infty(C,D)$, then the persistence curves are stable, i.e.*

$$\|P(C,\psi) - P(D,\psi)\| \leq \tilde{K} W_\infty(C,D). \tag{III.2}$$

where $\tilde{K} = \min\{KL^C + 2\Psi^D, KL^D + 2\Psi^D\}$

Proof. It follows directly from Theorem III.8. □

For example, if $\psi = \mathbf{le}$, the stability result due to [AGS18] can be retrieved from Corollary III.9.

Theorem III.10. *[AGS18] Let* C, D *be diagrams. Let* $L_{min} = \min\{L^C, L^D\}$*,* $L_{max} = \max\{L^C, L^D\}$ $N = \max\{n^C, n^D\}$ *and* $r_\infty(C,D) = \frac{2NW_\infty(C,D)}{L_{max}}$ *(this is called the relative error of* C *and* D*), then*

$$\|P(C, \mathbf{le}) - P(D, \mathbf{le})\|_1 \leq 2r_\infty(C,D)\left(\log[2r_\infty(C,D)] + L_{max}\frac{\log N}{N}\right).$$

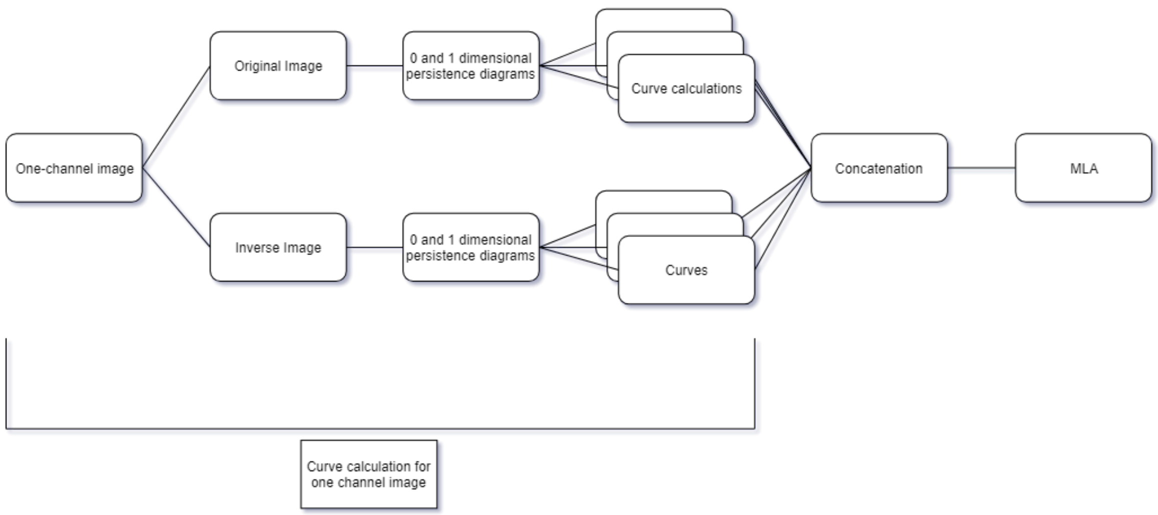


Figure III.5. PC Workflow for One Channel Image

Thusfar we have established the concept of a persistence curve, established a general bound, and established a class of persistence curves based on stable measurements that are robust with respect to noise.

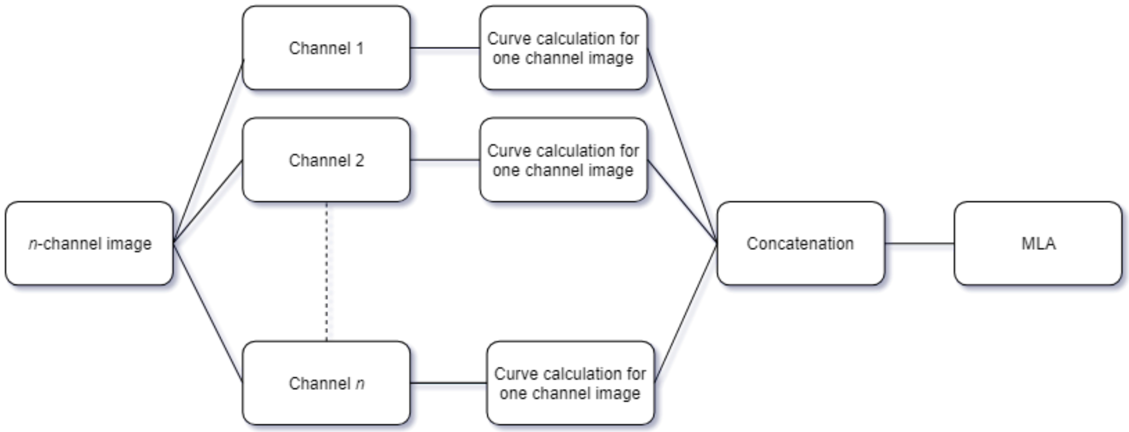
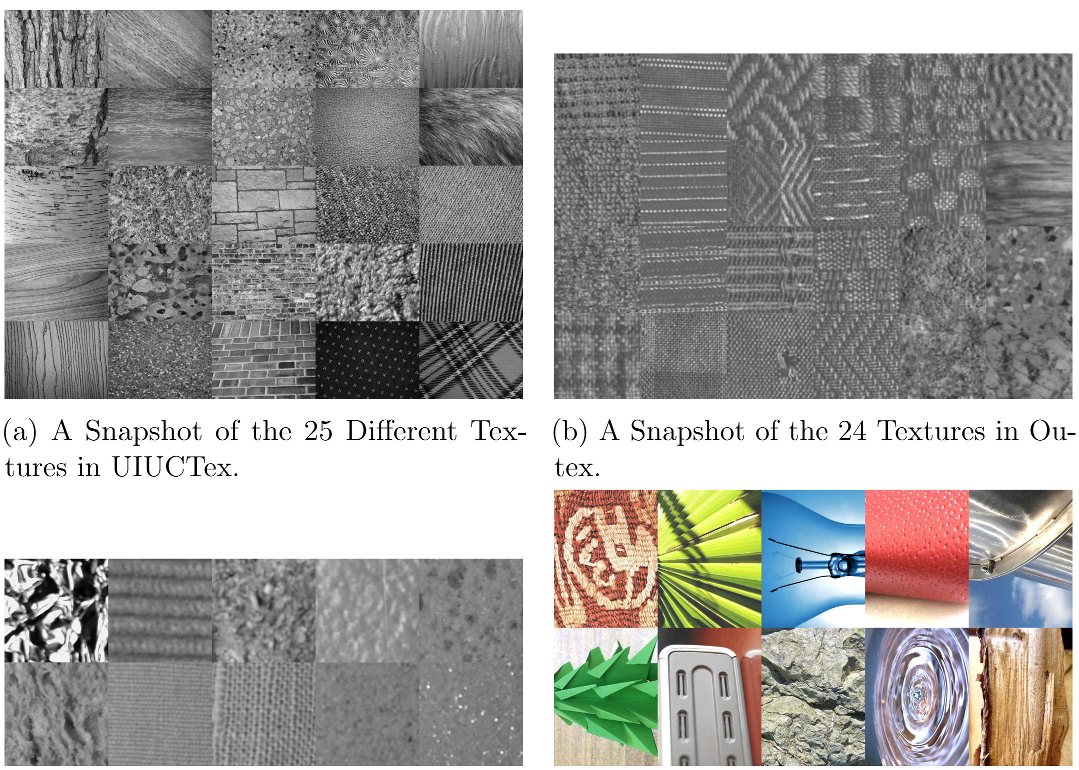


Figure III.6. PC Workflow for n-channel Image



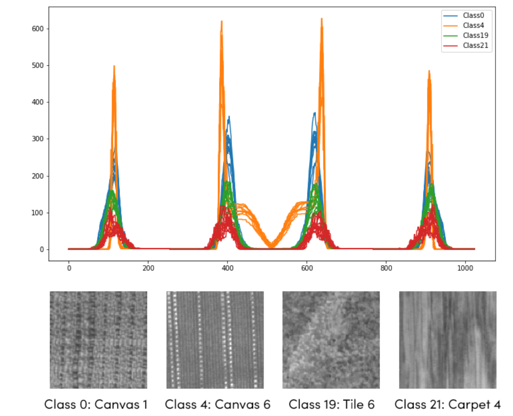
(a) A Snapshot of the 25 Different Textures in UIUCTex.

(b) A Snapshot of the 24 Textures in Outex.

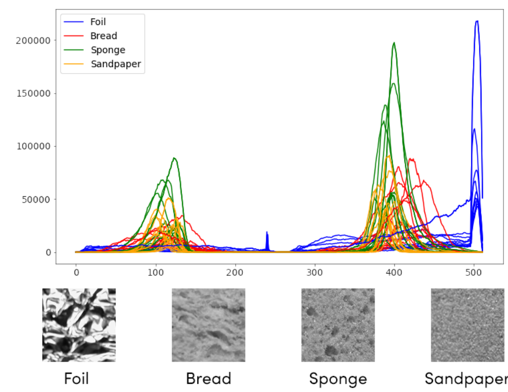
(c) A Snapshot of the 10 Textures of KTH-TIPS.

(d) A Snapshot of the 10 Textures in the FMD Database.

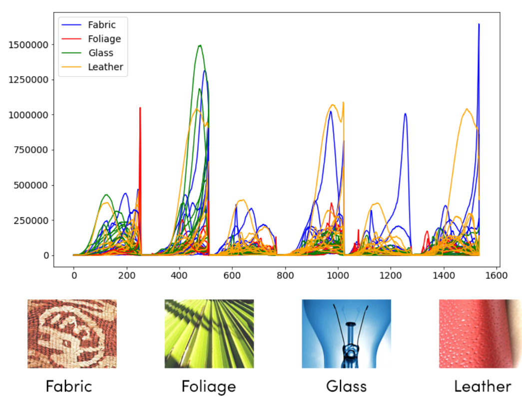
Figure III.7. Snapshots of the Texture Databases



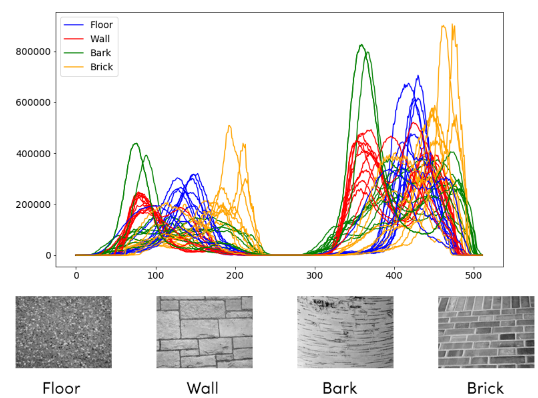
(a) Outex Betti Curves for Four Classes.



(b) KTH Betti Curves for Four Classes.



(c) FMD Betti Curves for Four Classes.



(d) UIUCTex Betti Curves for Four Classes.

Figure III.8. Curves for Selected Classes in Each Database

III.6. Applications to Texture Analysis

In this section, we apply PCs to texture classification. Texture analysis is one of the fundamental research areas in computer vision. One challenge in the area is to find intrinsic characteristics, or quantitative representations of textures in order to perform classifications, or employ statistics. PCs can be served as intrinsic characteristics of the textures, and this section is devoted to demonstrating effectiveness of PCs.

III.6.1. The Data

We consider 4 popular texture databases and describe each of them below.

(i) **Outex** is a database from the University of Oulu and consists of 15 test suites each with a different challenge [OMP+02]. We focus on the test suites 0, 3, and 10. Test suite 0 contains 24 texture classes with 20 images of each class that are 128 by 128 in size. The test suite 0 is equipped with 100 preset 50-50 train/test splits. A score on the test suite 0 is the average accuracy over all 100 splits. Test suite 3 is similar to 0 with the only difference in the misclassification costs. A score on the test suite 3 is the average weighted accuracy over the 100 splits. Finally, Test suite 10 tests rotational invariance (yu-min: Say more about it). In this suite there is a single train/test split of the 4320 images with the training set containing 1/9 of the images. A score on this test suite is a single accuracy score. See Figure III.7(b) for sample images of Outex.

(ii) **UIUC** dataset is a collection of textures from University of Illinois Urbana-Champagne [PNS12]. The dataset consists of 25 texture classes with 40 images of each texture. Each image is of size 640 by 480. A score on this set is the average accuracy of 100 random 50-50 train/test splits. See Figure III.7(a) for sample textures from UIUCTex.

(iii) **KTH-TIPS 2** [MTMF+] The KTH Textures under varying Illumination, Pose and Scale (TIPS) 2 is a database containing 81 200 by 200 grayscale images for each of its 10 textures. As the name suggests, each texture class contains images of different scales, rotations, and illuminations. A score on this set is the average accuracy of 100 random 50-50 train/test splits. See Figure III.7(c) for sample textures from KTH-TIPS 2.

(iv) **FMD** The Flickr Material Database (FMD) contains 100 RGB images of sizes 512 by 384 for each of its 10 materials [SRA14]. This database is the most challenging as it focuses more on material recognition, i.e. the classes contain images of many different objects at different scales. A score on this set is the average accuracy of 100 random 50-50 train/test splits. See Figure III.7(d) for sample textures from KTH-TIPS 2.

III.6.2. Classification

In order to illustrate the effectiveness of PCs, we keep our classification pipeline and model simple. Overall speaking, for each image, we calculate persistence diagrams, compute persistence curves, and feed PCs as features into machine learning algorithms (MLA), as shown in Figure III.5. We use two MLAs in this article—support vector machine (SVM) and random forest (RF). Since the intensity values of images range from 0 to 255, each PC is a vector in $\mathbb{R}^{256}$ associated with the standard Euclidean distance. For each image, there are two persistence diagrams, D_0 and D_1. We find it useful to consider the image and its complement. More precisely, consider the image $I^C(i,j) := 255 - I(i,j)$. Topologically, taking both I and I^C into account is similar to consider both sub-level and super-level sets of I. As shown in Figure III.1, this example shows that I^C may carry more information than single I may do.

Table III.3. Performance on FMD

Curves/MLA	SVM	RF
$(\boldsymbol{\beta}, \boldsymbol{\beta}\mathbf{S})$	31.8	38.6
$(\mathbf{le}(I),\mathbf{le}(I^C))$	34.5	41.7
$(\mathbf{mul}(I), \mathbf{mul}(I^C))$	34.1	39.2
$(\mathbf{le}(I),\mathbf{le}(I^C),\mathbf{mule}(I),\mathbf{mule}(I^C),\mathbf{mle}(I),\mathbf{mle}(I^C))$	38.8	41.3
$(\mathbf{le}(I),\mathbf{le}(I^C),\mathbf{mle}(I),\mathbf{mle}(I^C))$	37.1	39.7
$(\mathbf{le}(I), \mathbf{le}(I^C), \mathbf{mule}(I), \mathbf{mule}(I^C))$ + PS ([CHLSss])	33.2	44.5
Persistence Landscapes (PL)	30.1	32.9
(PL,$\mathbf{le}(I), \mathbf{le}(I^C), \mathbf{mule}(I), \mathbf{mule}(I^C)$)	25.7	41.8
(PS,PL,$\mathbf{le}(I), \mathbf{le}(I^C), \mathbf{mule}(I), \mathbf{mule}(I^C)$)	33.6	43.4
(PS,PL)	33.4	42.9

As examples, we illustrate the Betti curves on images and their complements for 4 different textures in the databases in Figures III.8a, III.8d, III.8b, and III.8c. From each figure, we observe that Betti curves of the same type of textures are of a similar pattern. For instance, Figure III.8a shows a sample of 4 textures from Outex. 0 to 255 is $\beta_0(I)$ curve; 256 to 511 is $\beta_1(I)$; 512 to 767 is $\beta_0(I^C)$; 768 to 1023. At this point, since we see that PCs can served as characteristics of textures, they will be the main features in our classification model.

Table III.4 compares the performance of various persistence curves, the Euler Characteristic Curve, and other TDA methods from the literature on the 4 databases. Up to our best knowledge, those TDA methods were applied to Outex 0; one of them were applied to UIUCTex and KTH.

For our applications we consider PCs described in Table III.1, and their combinations. There are five different sets of features in Table III.4 which consist of ECC, $\boldsymbol{\beta}$, le, mul, mule, and mle. The first two sets (Row 1 and 2 in Table III.4 do not require persistent homology. As discussed in [RW14], Betti numbers and ECC are useful descriptors.

First, we focus on the results of the Outex. We compared our methods with those developed in [GMBB18, RHBK15, LOC14, CCO17, CNO18] where they applied their TDA methods to Outex 0. Those performances were directly from their work [PC14]. From Table III.4, we see that ECC and Betti curves perform well on Outex, but not so well on UIUCTex or KTH. One possible explanation for this is these two curves are less stable than, say, the entropy curves. Hence, with the introduction of variation within a particular class (as in UIUC and KTH), performance is hindered. We also see that the Betti curve outperforms ECC. This makes sense as the Betti curves do not lose information through addition and subtraction as ECC does. We see once we consider the entropy curves, not only do we have high performance, we also have consistency across these data sets. PC falls slightly short of the Outex benchmark. We must note that the Outex benchmarks vary between the test suites. For Outex 0 and 3, the benchmark uses Gabor filtering [FS12]. With Outex 10, the benchmark model is a Local Binary Pattern with Variance [OPM02]. In this table we also see other TDA Methods. Sparse-TDA and kernel-TDA are both kernel methods. CLBP-SMC is also a kernel method that utilizes the Local Binary Patterns method. The Sliced Wasserstein Kernel, k_{PSS}, is another kernel method that proves to perform best out of the TDA methods on Outex 0. Another kernel method, EKFC+LMNN is a vectorization method that computes a descriptor based on the topology of a klein bottle, then combines this set of vectors with a metric learned through Large Margin Nearest Neighbors [PC14]. This method is able to achieve high scores on UIUC and KTH. We see Persistence Curves perform on a level similar to these high performing TDA methods, with the added benefit of their simplicity and generalized framework.

When we evaluate on more complex databases, such as UIUCTex, we see the scores begin to favor the entropy curves. Examining these texture databases reveal more variation of the images within each respective class. These scores suggest that there is a trade-off between the variation within a class and the performance of the more stable curves.

III.6.3. Combination of Persistence Statistics

Among these 4 databases, the FMD is the most challenging one. Unlike the other databases we consider, the FMD consists of RGB color images of 10 different materials. Within each class these materials will have different colors and even different textures. Our main task is to find characteristics capable of distinguishing between these materials. To do this, we treat each color channel (R, G, and B) as a single gray scale image. We generate our chosen persistence curves as shown in Table III.3 on each of these channels and their complements. Among PC features, the best classification performance by SVM is 38.8% (with entropy curves) as shown in Table III.3. The best classification performance by RF uses the same model and achieves a 41.3% recognition rate. This result is comparable to those in [LSAR10].

To improve to classification performance, we consider *persistence statistics*(PS) [CHLSss]. For each of the 12 diagrams (0 and 1 dimensional for each of the 3 channels and their inverses) we construct the datasets of death plus birth and lifespan. On each of these sets we calculate the mean, median, standard deviation, skewness, kurtosis, the 10th, 25th, 75th, 90th percentiles. Each of these values are concatenated to our persistence curves. Because we are combining both local and global features for these persistence diagrams, the concern for weighting the importance of these values arises. For this reason we turn to the machine learning algorithm known as random forests (RF). In short, a random forest is an ensemble method that combines a (possibly) large number of decision trees. This method allows us to forgo the weighting of these global values. Table III.3 reveals the performance FMD. We used a variety of methods including the use of SVM and also the use of Persistence Landscapes [Bub15]. This model was constructed by averaging the first 6 landscapes evaluated at the values 0,1,... 255. We see immediately that RF outperforms SVM in each scenario. Moreover, we again see that the combination of the entropy curves perform well and are only improved by the addition of persistence statistics. With the addition of Persistence Statistics, we see a decrease in the SVM recognition rate due to weighting issues. However, we see that RF has an increased recognition rate of 44.5%. We note this score is achieved using raw images and without any weight or parameter/hyper-parameter tuning. Persistence curves not only enjoy simplicity of concept and implementation, but also enjoy high performance potential.

III.6.4. Efficiency and Limitations

The computational efficiency of persistence curves depends on the chosen curve, the number of points in a given diagram, and the number of points at which the curve is evaluated. From the Definition III.2, we observe that the complexity of computing PC is roughly linear in both threshold and number of generators. The first experiment is to confirm this observation. For experimentation purposes, we examine the **mule**. We generate random diagrams in the following way. Each birth value is sampled from a uniform distribution on the interval 0 to 100. To each birth value b, we assign a death value by sampling a uniform distribution on the interval b to 101. We perform two experiments. First, we fix 1000 equidistant points in the interval 0 to 100. For each n in $\{10, 50, 100, 500, 1000, 5000, 10000, 50000, 100000, 500000, 1000000\}$ we generate 100 random diagrams with n points in the method described above.

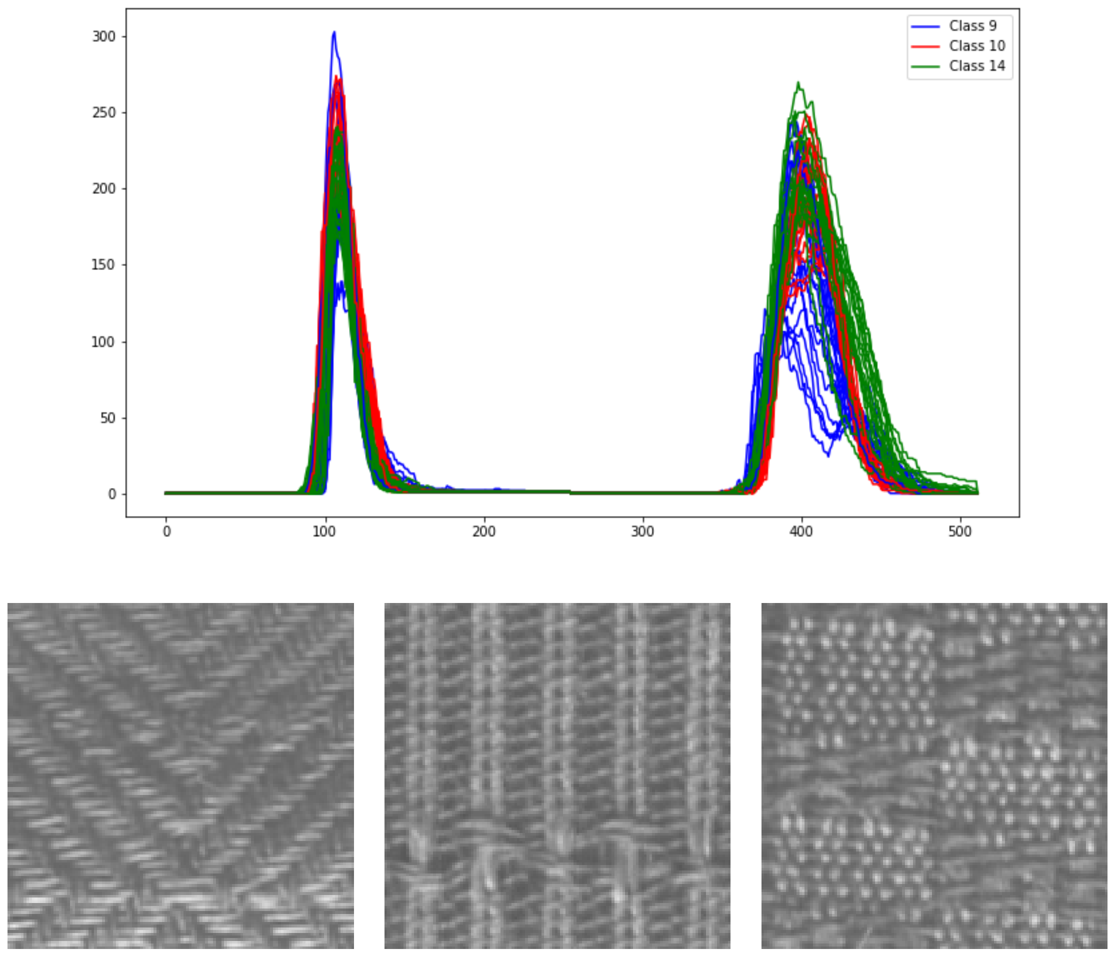


Figure III.9. Most Frequently Misclassified Classes of Outex 0.

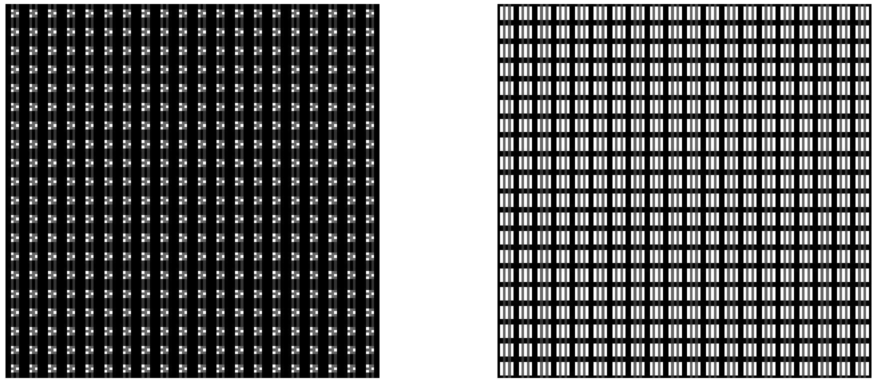


Figure III.10. Two Figures with Different Textures but Same Persistence Diagrams.

From here, we measure the average time taken to calculate the **mule** curve on each of these 100 diagrams at the 1000 equidistant points. In the second experiment we fix the number of points in a generated diagram at 1000. Then for each m in $\{10, 50, 100, 500, 1000, 5000, 10000, 50000\}$ we generate 100 random diagrams each with 1000 points. We calculate the average time to calculate the **mule** curve at m equidistant points on the interval 0 to 100. The results of these experiments appear in Tables III.6 and III.5 as well as Figure III.11. We see experimentally, the computation of persistence curves is quite efficient with linear complexity in both experiments.

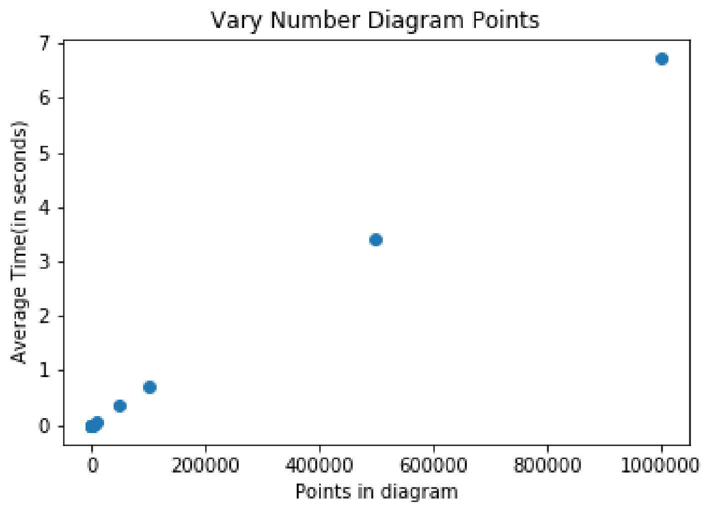
It is important to note that these persistence curves have some limitations. Because diagrams are not unique to a particular space, different textures may have similar persistence diagrams. This inverse problem is a challenging problem, and is a new research area in TDA [Cur19, OS18]. To illustrate this, we generate two images as shown in Figure III.10. To the human eye, these look like images of two different textures, but they actually produce the exact same persistence diagrams. While it is unlikely that real textures will produce exactly the same diagram, it is possible for different textures to produce similar diagrams hence similar persistence curves. For example in Figure III.9 we see 3 different classes of canvas texture in Outex 0. These 3 classes most often confused the classifiers. The rotational and size invariance of the topological descriptors play a role in this confusion as we see these textures have similar patterns in different sizes and orientations.

Table III.4. Comparison of Scores on the Databases

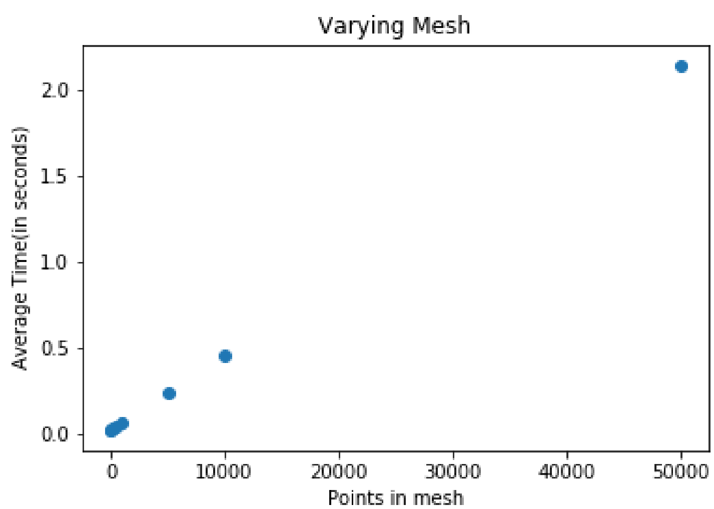
Proposed Features	**Outex0**	**Outex3**	**Outex10**	**UIUC**	**KTH**
ECC(I)	96.7	96.6	96.2	70.2	75.5
$[\boldsymbol{\beta}(I), \boldsymbol{\beta}(I^C)]$	97.9	98.0	97.2	80.8	78.3
[**le**(I),**le**(I^C)]	96.6	96.8	96.6	85.4	87.5
[**mul**(I), **mul**(I^C)]	96.2	96.0	96.0	83.6	82.2
[**le**(I),**le**(I^C),**mule**(I),**mule**(I^C),**mle**(I),**mle**(I^C))	97.7	97.9	97.2	88.8	90.8
Outex Benchmark	99.5	99.5	97.9	-	-
Other TDA methods	**Outex 0**	**Outex 3**	**Outex 10**	**UIUC**	**KTH**
sparse-TDA[GMBB18]	66.0	-	-	-	-
Kernel-TDA[RHBK15]	69.2	-	-	-	-
CLBP-SMC[LOC14]	87.5	-	-	-	-
k_{PSS} [CCO17]	98.8	-	-	-	-
Persistence Paths [CNO18]	97.8	-	-	-	-
EKFC+LMNN[PC14]	-	-	-	91.23	94.77

Table III.5. Computation Time for Varying Number of Points in Mesh

Mesh points	100	500	1000	5000	10000	50000
Time (in seconds)	0.026	0.045	0.068	0.240	0.452	2.141



(a) Computation Time for Varying Number of Points in Diagram



(b) Computation Time for Varying Number of Points in Mesh

Figure III.11. Computation Time Experiment

Table III.6. Computation Time for Varying Number of Points in Diagram

Diagram points	1000	5000	10000	50000	100000	500000	1000000
Time (in seconds)	0.007	0.034	0.068	0.376	0.719	3.415	6.735

III.7. Generalization and Conclusion

Persistence curves provide a simple general framework from which we can construct usable models for data analysis that retain the topological information contained in the persistence diagrams they are calculated from. In addition, these curves are compatible with machine learning, they are stable, they are efficient to compute, and by choice of function and statistic, one can alter the importance of points in different regions of the persistence diagrams. We have also shown that these curves create useful classifiers for texture analysis. The theory and experimentation presented here are by no means complete.

Open Questions. We present some open questions in no particular order.

Q1 Like Persistence Landscapes, what other vectorizations can be viewed as a special case of the Persistence Curves?

Q2 What conditions on the function ψ or the statistic T can lead to a more general and useful stability result?

Q3 Is there a statistical framework to perform "curve selection" that will produce an optimal or near optimal set of curves for modeling?

Q4 Could weighting be used to improve performance? In particular, is there a nice way to combine the local Persistence Curves with the global Persistent Statistics?

Q5 In what other areas might Persistence Curves be useful?

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